Math 200a Fall 2012 Exam 1

October 31, 2012

Do as many of the problems as well as you can; the exam may be too long for you to finish. You may use major theorems proved in class or the textbook, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Generally try to avoid quoting results proved only in homework exercises.

NAME:

Problem 1/ 10	
Problem 2 /10	
Problem 3 /10	
Problem 4 /10	
Total /40	

1. (10 pts)

For a prime p and a finite group G, let $O_p(G)$ be equal to the intersection of all Sylow-p-subgroups of G.

(a). Prove that $O_p(G)$ is the unique largest normal p-subgroup of G. In other words, prove that $O_p(G)$ is a normal p-subgroup, and that every normal p-subgroup of G is contained in $O_p(G)$.

(b). Find, with justification, $O_2(S_n)$ explicitly, for each $n \geq 4$. (The answer may depend on n.)

2. (10 pts)

(a). Let G be a finite group, and let H be a normal subgroup of G such that $|G : H| = p$ is prime. Suppose that K is any subgroup of G. Show that either $K \subseteq H$, or else $|K : K \cap H| = p$.

(b). Suppose in the situtation of part (a) that G acts on a finite set X. Then we can restrict the action to H, so that H also acts on the same set X. Show that given an orbit $\mathcal{O}$ of the G-action, either $\mathcal{O}$ is also an orbit of the H-action, or else $\mathcal{O}$ is a disjoint union of p H-orbits, all of the same size.

3. (10 pts)

Suppose that G is a group of order 56. Let P be a Sylow 2-subgroup of G, and Q a Sylow 7-subgroup.

(a). Show that either P is normal in G or else Q is normal in G.

(b). Suppose that Q is normal in G. Show that G is isomorphic to a semidirect product $Q \rtimes_\psi P$. Then show that either G is abelian, or else G contains a subgroup isomorphic to D_{14}, the dihedral group of order 14.

4. (10 pts) Let G be the subgroup $\langle R, S \rangle$ of $\mathrm{GL}(2, \mathbb{R})$ generated by the matrices R and S, where

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and where θ is a real number such that θ/π is irrational.

Prove that G has the presentation $\langle a, b \mid b^2 = e, ba = a^{-1}b \rangle$.