Math 142B Spring 2014 Exam 2- Solutions

1. Let $F : (-1, \infty) \to \mathbb{R}$ be the function defined by $F(x) = \int_0^x \ln(1+t)\,dt$.

(a) Consider the third Taylor polynomial $p_3(x)$ of the function $F(x)$ about $x_0 = 0$. Show that

$$p_3(x) = \frac{x^2}{2} - \frac{x^3}{6}.$$

Proof. Note that $F(0) = \int_0^0 \ln(1+t)\,dt = 0$. By the Fundamental Theorem of Calculus, $F'(x) = \ln(1+x)$. Differentiating thrice more yields $F''(x) = 1/(1+x)$, $F'''(x) = -1/(1+x)^2$, and $F^{(4)}(x) = 2/(1+x)^3$. Hence,

$$p_3(x) = \sum_{k=0}^{3} \frac{F^{(k)}(0)}{k!} x^k = \frac{x^2}{2} - \frac{x^3}{6}.$$

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(b) Prove that $F(x) > \frac{x^2}{2} - \frac{x^3}{6}$ for all $x > 0$.

Proof. By the Lagrange Remainder Theorem, for each $x > 0$, there exists $c \in (0, x)$ such that

$$F(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{F^{(4)}(c)}{4!} x^4 = \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{12(1+c)^3} x^4.$$

Since $c > 0$, $(1+c)^3 > 0$ and $x^4 > 0$ for all x. Hence, the remainder term is positive so

$$F(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{12(1+c)^3} x^4 > \frac{x^2}{2} - \frac{x^3}{6}.$$

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2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is infinitely differentiable and which satisfies $f'(x) = 2f(x)$ for all $x \in \mathbb{R}$. For any $r > 0$ show that the Taylor series expansion of f about $x_0 = 0$ converges to $f(x)$ for all $x \in [-r, r]$. Conclude that the Taylor series expansion of f about $x_0 = 0$ converges to f for all real numbers x.

Proof. Let $c = \max_{[-r,r]} f'(x)$. Let $M = \max\{c, 2\}$.We wlil prove that for all $n \in \mathbb{N}$ and all $x \in [-r, r]$, that $|f^{(n)}(x)| \leq M^n$, so by Theorem 8.14, the Taylor series converges on $[-r, r]$.

We proceed by induction. For the base case, $|f'(x)| \leq c \leq M$ for all $x \in [-r, r]$. Now by induction

$$|f^{(n)}(x)| = 2|f^{(n-1)}(x)| \leq 2M^{n-1} \leq M^n$$

since $M \geq 2$. Hence, $|f^{(n)}(x)| \leq M^n$ for all n and all $x \in [-r, r]$. Since for any x, there exists an r such that $r > |x|$, the Taylor series expansion converges for all $x \in \mathbb{R}$. ■

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions and fix $x_0 \in I$. Assume f and g are infinitely differentiable.

(a) Suppose that f and g have infinite order of contact at x_0 and that f and g are both analytic at x_0. Prove that there is an open neighborhood I of x_0 such that $f(x) = g(x)$ for all $x \in I$.

Proof. Since f is analytic, there exists an open neighborhood (a, b) of x_0 where f is equal to its Taylor series. Similarly, there exists (c, d) an open neighborhood of x_0 where g equals its Taylor series.

Let $I = (\max\{a, c\}, \max\{b, d\}) = (a, b) \cap (c, d)$. This is an open neighborhood of x_0 where

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(x_0)}{k!}(x - x_0)^k.$$

But since f and g have infinite order of contact, their Taylor series agree, so $f(x) = g(x)$ for all $x \in I$. ∎

(b) Suppose that f and g have infinite order of contact at x_0, but they are not necessarily analytic at x_0. Must there be an open neighborhood I of x_0 such that $f(x) = g(x)$ for all $x \in I$? Prove or give an explicit counterexample with explanation.

Proof. As a counterexample, let

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and let $g(x) = 0$ be identically zero. Then f and g have infinite order of contact (by theorems in section 8.6). But $f(x) \neq 0$ for any $x \neq 0$, so there is no open neighborhood wher $f(x) = g(x)$. ∎