MATH 142B SPRING 2014 MIDTERM 1 - SOLUTIONS

1. (10 pts) Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are bounded functions such that $f(x) \le g(x)$ for all $x \in [a, b]$.

(a). (5 pts) Prove that for any two partitions P and Q of $[a, b]$, we have $L(f, P) \le U(g, Q)$.

Proof. Let $P = \{x_0, \dots, x_n\}$. Since $f(x) \le g(x)$ for all $x \in [a, b]$, on each partition interval of P, we have $\inf_{x \in [x_{i-1}, x_i]} f(x) \le \inf_{x \in [x_{i-1}, x_i]} g(x)$. Hence, $L(f, P) \le L(g, P)$. By the Refinement Lemma, $L(g, P) \le L(g, P \cup Q)$. By equation (6.3) in the text, $L(g, P \cup Q) \le U(g, P \cup Q)$. Finally, again by the Refinement Lemma, $U(g, P \cup Q) \le U(g, Q)$. We conclude that $L(f, P) \le U(g, Q)$.

(b). (5 pts) Use part (a) to prove that $\underline{\int_a^b} f \le \overline{\int_a^b} g$.

Proof. Fix a partition P of $[a, b]$. Then by part (a), we have $L(f, P) \le U(g, Q)$ for any partition Q of $[a, b]$. Hence, $L(f, P) \le \inf\{U(g, Q) \mid Q\} = \overline{\int} g$ (by closedness of the set $[L(f, P), \infty)$, each $U(g, Q)$ is in this set, so the infimum must also be). But P was arbitrary so we have $L(f, P) \le \overline{\int} g$ for any partition P. Hence, $\sup L(f, P) = \underline{\int} f \le \overline{\int} g$ by closedness of $(-\infty, \overline{\int} g]$.

2. (10 pts) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $\delta, \epsilon > 0$ are given such that for all $u, v \in [a, b]$ with $|u - v| < \delta$, one has $|f(u) - f(v)| < \epsilon$.

Suppose that $P = \{x_0, \dots, x_n\}$ is a partition of $[a, b]$ with the property that $\operatorname{gap} P < \delta$.

Prove that $U(f, P) - L(f, P) \le \epsilon(b - a)$. (Recall that $\operatorname{gap} P = \max\{(x_i - x_{i-1}) | 1 \le i \le n\}$.)

Proof. On each partition interval $[x_{i-1}, x_i]$ f is continuous so by the extreme value theorem there exist points y_i, and z_i where f achieves a minimum and a maximum on the interval. Since $y_i, z_i \in [x_{i-1}, x_i]$, we have $|y_i - z_i| < \delta$. So by hypothesis, $f(z_i) - f(y_i) < \epsilon$ Hence,

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} [f(z_i) - f(x_i)](x_i - x_{i-1}) \le \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) = \epsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon(b - a).$$

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3. (10 pts) Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that if f is integrable, then $\int_a^b f = 0$.

Proof. For any partition P of $[a, b]$, we have $L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ where m_i is the infimum of f in the interval $[x_{i-1}, x_i]$. By the density of the rationals, in each partition interval there exists a point y_i such that $f(y_i) = 0$. Hence, $m_i \leq 0$ for all i. So $L(f, P) \leq 0$ for all partitions P. Thus, $\sup\{L(f, P) \mid P\} = \underline{\int} f \leq 0$.

Similarly, we have $\overline{\int} f \geq 0$. Now since f is integrable, $\underline{\int} f = \overline{\int} f = \int f$. So we have $\int f \leq 0$ and $\int f \geq 0$ which is only possible if $\int f = 0$.