Math 142B
Final Examination
September 2, 2011
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Instructions

1. **You may use any type of calculator, but no other electronic devices during this exam.**
2. **You may use one page of notes, but no books or other assistance during this exam.**
3. **Write your *Name*, *PID*, and *Section* on the front of your Blue Book.**
4. **Write your solutions clearly in your Blue Book**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order they appear in the exam.**
   (c) **Start each question on a new page.**
5. **Read each question carefully, and answer each question completely.**
6. **Show all of your work; no credit will be given for unsupported answers.**

1. Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is not integrable on $[0,1]$.

2. Let $f : [a,b] \to \mathbb{R}$ be continuous. Use the First Fundamental Theorem to prove that

$$\frac{d}{dx}\left[\int_x^b f\right] = -f(x) \quad \text{for all } x \in (a,b).$$

(Note: This result motivates the definition $\int_d^c f = -\int_c^d f$ for $c < d$ in $[a,b]$. You must, of course, prove the result without using this definition.)

3. Let $f : [a,b] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } a \le x < \frac{a+b}{2}, \\ 1 & \text{if } \frac{a+b}{2} \le x \le b. \end{cases}$$

Show that there is no point $x_0 \in [a,b]$ at which $f(x_0) = \dfrac{1}{b-a}\displaystyle\int_a^b f$. Explain why this does not contradict the Mean Value Theorem for Integrals.

4. Prove that $1 + \frac{1}{2}x - \frac{1}{8}x^2 < \sqrt{1+x} < 1 + \frac{1}{2}x$ for every $x > 0$.

5. Let $p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$, the n^{th} Taylor polynomial for $\ln(1+x)$.
   Prove the following inequalities.

   (a) $|\ln(1+x) - p_n(x)| \leq \frac{1}{1+x} \cdot \frac{|x|^{n+1}}{n+1}$ if $-1 < x \leq 0$

   (b) $|\ln(1+x) - p_n(x)| \leq \frac{x^{n+1}}{n+1}$ if $0 \leq x \leq 1$

6. For each natural number n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = e^{-nx^2}.$$

   (a) Determine the function $f : \mathbb{R} \to \mathbb{R}$ that $\{f_n\}$ converges to pointwise.

   (b) Prove that the convergence is not uniform.

7. Exhibit an example of a sequence of differentiable functions $f_n : (-1,1) \to \mathbb{R}$ that converges uniformly but for which $f'(0)$ is unbounded (that is, $\lim_{n\to\infty} |f_n'(0)|$ diverges to ∞).