Math 142B Practice Midterm (Eggers 2011) - Solutions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define

$$G(x) = \int_0^x (x-t)f(t)\,dt \text{ for all } x.$$

Use the Second Fundamental Theorem to show that $G''(x) = f(x)$ for all x. (Hint: Use the linearity property of the integral to rewrite it in a more convenient form.)

Proof. By the linearity of the integral,

$$G(x) = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt.$$

Now by the Second Fundamental Theorem,

$$G'(x) = \frac{d}{dx}\left[x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt\right] = xf(x) + \int_0^x f(t)\,dt - xf(x) = \int_0^x f(t)\,dt.$$

Applying the Second Fundamental Theorem once more yields:

$$G''(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x).$$

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2. Let $f(x) = e^x$. We have seen that the n^{th} Taylor polynomial for f at $x = 0$ is given by

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n.$$

Prove that for every real number x, $f(x)$ is equal to its Taylor series at $x = 0$.

Proof. Let $x_0 \in \mathbb{R}$. Choose $r > |x_0|$. We will show that $f(x)$ equals its Taylor series on the interval $[-r, r]$ and so f equals its Taylor series at x_0. By Theorem 8.14, it suffices to choose an $M > 0$ such that for all $x \in [-r, r]$, $|f^{(n)}(x)| \le M^n$.

We note that $f^{(n)}(x) = e^x$ for all n, and that e^x is a strictly increasing function (as it has strictly positive derivative everywhere). Hence, if we choose $M = e^r$, then for all $n \in \mathbb{N}$, and all $x \in [-r, r]$,

$$|f^{(n)}(x)| \le \max_{[-r,r]} e^x = e^r \le M^n.$$

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3. Use the Lagrange Remainder Theorem to show that

$$0 < x - \ln(1+x) < \frac{1}{2}x^2 \text{ for all } x > 0.$$

Proof. Let $f(x) = \ln(1+x)$. First, we note that by rearranging inequalities, it suffices to show that

$$x - \frac{1}{2}x^2 < \ln(1+x) < x.$$

We compute the first three derivatives of $\ln(1+x)$:

$$f'(x) = \frac{1}{1+x} \quad \text{and} \quad f''(x) = -\frac{1}{(1+x)^2} \quad \text{and} \quad f'''(x) = \frac{1}{(1+x)^3}$$

So the first and second Taylor polynomials for $\ln(1+x)$ at $x = 0$ are

$$p_1(x) = x$$

$$p_2(x) = x - \frac{1}{2}x^2.$$

Now by Lagrange Remainder Theorem, there exists a c between 0 and x such that

$$\ln(1+x) = x + \frac{f''(c)}{2!}x^2.$$

But since $f''(x) < 0$ for all $x > 0$, it follows that the remainder term is always negative and hence $\ln(1+x) < x$. Similarly, there exists a c such that

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{f'''(c)}{3!}x^3.$$

Since $f'''(x) > 0$ for all $x > 0$, here the remainder term is always positive so

$$x - \frac{1}{2}x^2 < \ln(1+x) < x.$$

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4. Let $f : \mathbb{R} \to \mathbb{R}$ have derivatives of all orders and satisfy

$$f'(x) = f(x) \text{ for all } x,$$

$$f(0) = 2.$$

(a) Find a formula for the coefficients of the n^{th} Taylor polynomial for f at $x = 0$.

Proof. By induction, $f^{(k)}(x) = f(x)$ for all k and all x. Hence, the coefficient for the kth term in the Taylor polynomial is given by

$$\frac{f^{(k)}(0)}{k!}x^k = \frac{f(0)}{k!}x^k = \frac{2}{k!}x^k.$$

So we have

$$p_n(x) = \sum_{k=0}^{n} \frac{2}{k!}x^k.$$

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(b) Show that the Taylor series for f at $x = 0$ converges for all x.

Proof. Again, we show convergence of the Taylor series at an arbitrary $x_0 \in \mathbb{R}$. Let $r > |x_0|$. We again use Theorem 8.14 to show that the Taylor series converges on $[-r, r]$. Since $f(x)$ is differentiable, therefore it is continuous. By the Extreme Value Theorem, there exists some $M = \max_{[-r,r]} f(x)$. Now, for all $n \in \mathbb{N}$ and all $x \in [-r, r]$,

$$|f^{(n)}(x)| = |f(x)| \leq M \leq M^n$$

(since $M \geq 2$). ■