Name: ______________________________ **PID:** ____________________

Math 140A: Final Exam
Foundations of Real Analysis

- You have 3 hours.
- No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- You may not use homework problems (without proof) in your solutions.

1. (10 points) Let J be the set of all positive integers. Let A be an infinite set.

(a) (5 points) Prove that there exists a 1-1 function $f : J \to A$.

(b) (5 points) Prove that the set of all 1-1 functions $f : J \to A$ is uncountable.

2. (10 points) Let X be a nonempty set. For $x \in X$ and $y \in X$, define

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

(a) (3 points) Prove that d is a distance function.

(b) (3 points) Prove that if X is connected, then X has exactly one element.

(c) (4 points) Prove that if X is compact, then X is finite.

3. (10 points) Let $\{x_n\}$ be a sequence of real numbers. Assume that the "even" and "odd" subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are convergent. Denote $a = \lim_{n\to\infty} x_{2n}$ and $b = \lim_{n\to\infty} x_{2n+1}$.

(a) (5 points) Prove that if $a \neq b$, then the sequence $\{x_n\}$ is not convergent.

(b) (5 points) Prove that if $a = b$, then the sequence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = a$.

4. (10 points) Let $\{x_n\}$ be a bounded sequence of real numbers. Denote $\alpha = \limsup_{n\to\infty} x_n$. Define a new sequence $\{y_m\}$ by letting $y_m = \sup\{x_n | n \geqslant m\}$, for every $m \geqslant 1$.

(a) (5 points) Prove that the sequence $\{y_m\}$ is monotonically decreasing and convergent.

(b) (5 points) Prove that $\lim_{m\to\infty} y_m = \alpha$.

5. (10 points)

(a) (5 points) Prove that the series $\sum \frac{n^3}{3^n}$ converges.

(b) (5 points) Let $\{a_n\}$ be a sequence of real numbers such that $a_1 \geqslant a_2 \geqslant a_3 \geqslant ... \geqslant 0$. Assume that $3a_{2n} \leqslant a_n$, for all $n \geqslant 1$. Prove that the series $\sum a_n$ converges.

6. (10 points) Let $\{a_n\}$ be a sequence of real numbers.

(a) (5 points) Assume that the series $\sum a_n$ is absolutely convergent. Let $\{b_n\}$ be a bounded sequence of real numbers. Prove that the series $\sum a_n b_n$ is absolutely convergent.

(b) (5 points) Assume that the series $\sum a_n b_n$ is convergent, for any bounded sequence $\{b_n\}$ of real numbers. Prove that the series $\sum a_n$ is absolutely convergent.

7. Let A be a nonempty set of real numbers and let $f : A \to [0, \infty)$ be given by $f(x) = x^2$.

(a) (5 points) Prove that if A is bounded, then f is uniformly continuous.

(b) (5 points) Prove that if A is open and f is uniformly continuous, then A is bounded.

8. Let $\mathbb{R}$ denote the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that $f(x) = g(x)$, for every rational number x.

Prove that $f(x) = g(x)$, for every $x \in \mathbb{R}$.

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1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
5		out of 10 points
6		out of 10 points
7		out of 10 points
8		out of 10 points
Total		out of 40 points