MATH 140A FALL 2015 MIDTERM 2 SOLUTIONS

1 (10 pts). Decide if the following series converges or not. Justify your answer.

$$\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+5}$$

Solution. Since $n^2+5 > n^2$ for all n, we have $\dfrac{\sqrt{n}}{n^2+5} < \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{3/2}}$ for all $n \geq 0$. Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$, it converges. Thus $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+5}$ also converges by the comparison test.

2. (a) (7 pts). Let $\{p_n\}$ be a sequence in a metric space X. Prove that if $\{p_n\}$ converges, then the sequence is bounded. (This is a result in the text, so you must reproduce the proof here rather quoting that result.)

(b) (3 pts). Give an example of a bounded sequence in a metric space X that does not converge. Briefly justify your answer.

Solution. (a) Suppose $\{p_n\}$ converges to $p \in X$. Choose any $\epsilon > 0$; then by definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(p_n, p) < \epsilon$. If

$$r = \max(\epsilon, d(p_1, p), d(p_2, p), \ldots, d(p_{N-1}, p)),$$

then the open neighborhood $N_r(p)$ contains p_n for all $n \geq 1$. Thus $N_r(p)$ contains the range $\{p_n | n \geq 1\}$ of the sequence, and so $\{p_n\}$ is bounded by definition.

(b). Let $\{p_n\}$ be defined by $p_n = (-1)^n$ for $n \geq 1$, so p_n alternates between 1 and -1. If p is any real number, then either $d(p, 1) \geq 1/2$ or $d(p, -1) \geq 1/2$. Thus taking $\epsilon = 1/2$, it is impossible to choose $N \in \mathbb{N}$ such that $p_n \in N_\epsilon(p)$ for all $n \geq N$. So $\{p_n\}$ does not converge to p for any p.

3 (10 pts). Let $E = \{1/n \,|\, n = 1, 2, 3, \ldots\} \bigcup \{0\}$ in $\mathbb{R}$. Show that E is compact directly from the definition (not using the Heine-Borel Theorem). In other words, prove directly that any open cover has a finite subcover.

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Solution. Suppose $\{G_\alpha\}$ is an open cover of E. Then 0 is in one of the sets of the open cover, say $0 \in G_{\alpha_0}$. Since G_{α_0} is open, there is a neighborhood $N_\epsilon(0) = (-\epsilon, \epsilon) \subset G_{\alpha_0}$ for some $\epsilon > 0$. By the Archimedean property, there is N such that $N > 1/\epsilon$; then $0 < 1/n < \epsilon$ for all $n \geq N$ and so $1/n \in G_{\alpha_0}$ for all $n \geq N$. For each $1 \leq n \leq N-1$ we can choose some open set in the cover G_{α_n} with $1/n \in G_{\alpha_n}$. Thus E is contained in $\bigcup_{n=0}^{N-1} G_{\alpha_n}$ and thus the open cover of E has a finite subcover.

4 (10 pts). Let E and F be compact subsets of a metric space X. Prove that $E \cup F$ is compact.

Solution. Suppose that $\{G_\alpha\}$ is an arbitrary open cover of $E \cup F$. Then it is also an open cover of E and an open cover of F. Since E is compact, there is a finite subcover, that is, $E \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$ for some indices $\alpha_1, \ldots, \alpha_n$. Similarly, since F is compact, there is a finite subcover, say $F \subseteq G_{\beta_1} \cup \cdots \cup G_{\beta_m}$. Then $E \cup F \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup G_{\beta_1} \cup \cdots \cup G_{\beta_m}$. Thus the open cover $\{G_\alpha\}$ of $E \cup F$ has a finite subcover.

5 (10 pts). Let $\{p_n\}$ and $\{q_n\}$ be Cauchy sequences in the metric space X with distance function d. Prove that $\lim_{n\to\infty} d(p_n, q_n)$ exists.

Solution.

Let $\epsilon > 0$ be fixed. By definition we can choose $N_1 \in \mathbb{N}$ such that $d(p_m, p_n) < \epsilon/2$ for all $m, n \geq N_1$. Similarly we can choose $N_2 \in \mathbb{N}$ such that $d(q_m, q_n) < \epsilon/2$ for all $m, n \geq N_2$. If $N = \max(N_1, N_2)$ then for all $m, n \geq N$ we have

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) < d(p_m, q_m) + 2(\epsilon/2)$$

by the triangle inequality. Then $d(p_n, q_n) - d(p_m, q_m) < \epsilon$ for $m, n \geq N$. Interchanging m and n we also have $d(p_m, q_m) - d(p_n, q_n) < \epsilon$ and thus $|d(p_n, q_n) - d(p_m, q_m)| < \epsilon$ for $m, n \geq N$. Since the distance function in $\mathbb{R}$ is given by the absolute value of the difference, this shows that $\{d(p_n, q_n)\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, this Cauchy sequence converges and so $\lim_{n\to\infty} d(p_n, q_n)$ exists.