MATH 140A FALL 2015 MIDTERM 1– SAMPLE SOLUTIONS

1. (a) (5 pts). Carefully define the following:

(i). What it means for a set X with a distance function d to be a metric space.

(ii). What it means for $p \in X$ to be a limit point of a subset E of X.

(iii). The closure $\overline{E}$ of a subset E of a metric space X.

(b) (5 pts). Let E be a nonempty set of real numbers which is bounded below. Prove that $\inf E \in \overline{E}$.

(c) (5 pts). Let $\mathbb{Q}$ be the set of rational numbers in the metric space $\mathbb{R}$. What is $\overline{\mathbb{Q}}$? Justify your answer.

Solution.

(i). X is a metric space with distance function d if $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0$ if and only if $x = y$; $d(x, y) = d(y, x)$ for all $x, y \in X$; and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(ii). The point p is a limit point of E provided that for every $r > 0$, there is some $q \in E$ with $q \neq p$ such that $q \in N_r(p) = \{x \in X | d(x, p) < r\}$.

(iii). Let E' be the set of all limit points of E. Then we define $\overline{E} = E \cup E'$.

(b). Let $\alpha = \inf E$. If $\alpha \in E$ we are done, since $E \subseteq \overline{E}$. Thus suppose that $\alpha \notin E$. Given $r > 0$, then $\alpha + r$ is not a lower bound of E, by definition of the infimum. Thus there is $x \in E$ such that $\alpha \leq x < \alpha + r$, and since by assumption $\alpha \notin E$, we have $\alpha < x < \alpha + r$. This shows that $\alpha \neq x \in N_r(\alpha)$ and thus by definition α is a limit point of E. But then $\alpha \in E' \subseteq \overline{E}$.

(c). We claim that $\overline{\mathbb{Q}} = \mathbb{R}$. Let $\alpha \in \mathbb{R}$. Then for all $r > 0$, the interval $(\alpha, \alpha + r)$ must contain a rational number, by a theorem proved in class and in the book (Theorem 1.20). Thus for all $r > 0$ we have that $N_r(\alpha)$ contains a point in $\mathbb{Q}$ other than α and hence α is a limit point of $\mathbb{Q}$. Thus in fact $\mathbb{Q}' = \mathbb{R}$ and so certainly $\overline{\mathbb{Q}} = \mathbb{R}$.

2 (5 pts). Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in the Euclidean space $\mathbb{R}^k$. Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.$$

Solution. Recall that $|\mathbf{x}|^2 = (\mathbf{x}\cdot\mathbf{x})$, where $\cdot$ indicates the dot product of two vectors. Since the dot product is additive in each coordinate, we calculate that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}+\mathbf{y}\cdot\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y}\cdot\mathbf{x}-\mathbf{y})$$

$$= (\mathbf{x}\cdot\mathbf{x}) + (\mathbf{y}\cdot\mathbf{x}) + (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{x}) - (\mathbf{y}\cdot\mathbf{x}) - (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y})$$

$$= 2(\mathbf{x}\cdot\mathbf{x}) + 2(\mathbf{y}\cdot\mathbf{y}) = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.$$

3. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the natural numbers.

(a) (5 pts). Let A be the set of all functions $f : \mathbb{N} \to \{0,1\}$. Prove that A is uncountable directly by using Cantor's diagonal process (do not quote a theorem from the book).

(b) (5 pts). Let B be the set of all functions $f : \{0,1\} \to \mathbb{N}$. Is B countable or is it uncountable? Justify your answer.

Solution.

(a). Suppose that A is countable. Then we can enumerate the elements of A as $f_1, f_2, f_3, \dots$. Create a new function $g \in A$ where $g(n) = 0$ if $f_n(n) = 1$ and $g(n) = 1$ if $f_n(n) = 0$. By construction, g differs from f_n in its value at n. Thus $g \neq f_n$ for all $n \in \mathbb{N}$. Thus g is an element of A which is not among the enumerated sequence $f_1, f_2, \dots$, a contradiction. Thus A is uncountable.

(b). The set B is countable. An element of B is a function with two arbitrary values $f(0), f(1)$. Thus the set B is in one-to-one correspondence with ordered pairs of elements in $\mathbb{N}$, that is, with the cartesian product $\mathbb{N} \times \mathbb{N}$. But we proved that that such a cartesian product is countable (Theorem 2.13 in the text).

4. Let X be a metric space. Let E° denote the interior of a subset E of X. Suppose that E and F are subsets of X.

(a) (5 pts). Is it always true that $E^\circ \cap F^\circ = (E \cap F)^\circ$? Prove or give a counterexample.

(b) (5 pts). Is it always true that $E^\circ \cup F^\circ = (E \cup F)^\circ$? Prove or give a counterexample.

Solution.

(a). This is true. Recall that the interior E° is the set of points $x \in E$ such that there is $r > 0$ with $N_r(x) \subset E$. If $x \in (E \cap F)^\circ$, so there is $r > 0$ such that $N_r(x) \subset E \cap F$, then $N_r(x) \subset E$ and $N_r(x) \subset F$. Thus $x \in E^\circ$ and $x \in F^\circ$ and hence $x \in E^\circ \cap F^\circ$.

Conversely, if $x \in E^\circ \cap F^\circ$, Then there is $r_1 > 0$ such that $N_{r_1}(x) \in E$ and $r_2 > 0$ such that $N_{r_2}(x) \in F$. Let $r = \min(r_1, r_2) > 0$. Then $N_r(x) \subset E$ and $N_r(x) \subset F$, so $N_r(x) \subset E \cap F$. Thus $x \in (E \cap F)^\circ$.

(b). This is not always true. For example, let $X = \mathbb{R}$ and let $E = [0, 1]$ and $F = [1, 2]$. Then $E \cup F = [0, 2]$. So clearly $1 \in (E \cup F)^\circ$, but $1 \notin E^\circ$ and $1 \notin F^\circ$.

(Though this is not necessary to answer what was asked, we remark that one inclusion is always true however, namely $E^\circ \cup F^\circ \subset (E \cup F)^\circ$.)