Math 140a Fall 2015 Homework 6

Due Friday November 6 by 5pm in HW box in basement of AP&M

Reading

All references are to Rudin, 3rd edition.

Continue to Read Chapter 3. This homework contains problems about convergent sequences, upper and lower limits, and infinite series. Only the material in the text up to page 63 (up through the section on "Series of nonnegative terms") is needed to solve the problems about series.

Assigned problems from the text (write up full solutions and hand in):

Chapter 3: #4, 6(a)(b)(c), 7, 14(a)(b)

Suggestion for #7: Consider the terms where $a_n \leq 1/n^2$ and terms where $a_n > 1/n^2$ separately. In each case try to compare the terms $\sqrt{a_n}/n$ with terms of a known convergent sequence.

Additional problems (write up and hand in):

A. Let $\{s_n\}$ be a sequence of real numbers. Let $s^* = \limsup_{n\to\infty} s_n$ be the upper limit (or limit superior) of $\{s_n\}$. This is defined in the text to be the supremum of the set of subsequential limits of $\{s_n\}$. In Theorem 3.17 it is shown that s^* has the following property: Given any $s^* < x$, then there is $N \in \mathbb{N}$ such that $s_n < x$ for all $n \geq N$. In other words, given any number bigger than the upper limit, the sequence eventually gets below that number and stays below it. In this problem, we show that this idea can be used to give alternative ways to get s^* without using subsequential limits.

(a). Let $\{s_n\}$ be a sequence of real numbers. To simplify this problem, assume that $\{s_n\}$ is bounded, so that $s^* = \limsup_{n\to\infty} s_n \in \mathbb{R}$. Let B be the set of all real numbers x with the property that there exists $N \in \mathbb{N}$ such that $s_n < x$ for all $n \geq N$. Prove that $\inf B = s^*$ (you may assume Theorem 3.17(b) in your proof). Give examples showing that depending on the sequence $\{s_n\}$, one might have $s^* \in B$ or $s^* \notin B$.

(b). Again let $\{s_n\}$ be a bounded sequence of real numbers. For each $m \in \mathbb{N}$, define $t_m = \sup\{s_n | n \geq m\}$. In other words, we take a tail of the sequence beginning at some m and then take the supremum of the range of that tail. Show that $\{t_m\}$ is a bounded monotonic decreasing sequence, and that $\lim_{m\to\infty} t_m = s^*$. (By the way, this way of defining the upper limit explains the notation lim sup, since it shows that $s^* = \lim_{m\to\infty} \sup\{s_n | n \geq m\}$.)

B. Prove the claim in Example 3.18(c) in the text: If $\{s_n\}$ is a sequence of real numbers, then $\lim_{n\to\infty} s_n = s$ if and only if $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$. To simplify the problem, assume that $\{s_n\}$ is bounded and so s is a real number.

C. Let $\{a_n\}$ be a sequence of positive numbers such that $a_1 \geq a_2 \geq a_3 \geq \ldots$ and such that $a_{2n} \leq a_n/3$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

Optional problems (handing in is not required)

none this week; study for the exam.