Math 140a Fall 2015 Homework 2

Due Friday October 9 by 5pm in HW box in basement of AP&M

Reading

All references are to Rudin, 3rd edition.

Finish reading Chapter 1 and begin reading about countability and metric spaces in Chapter 2.

Assigned problems (write up full solutions and hand in):

Chapter 1: #11, 13, 14, 15, 17, 19

Problem not from the text (also to be handed in):

A. In this problem you fill in some of the details about expressing numbers as decimals, which are only briefly mentioned in the text in Section 1.22. You may find the following formula for the partial sum of a geometric series helpful:

$$(1 + a + a^2 + \cdots + a^n) = \frac{a^{n+1} - 1}{a - 1}, \quad \text{for any real } a \neq 1.$$

(a). Suppose that $n \in \mathbb{Z}$ is an integer and $a_1, a_2, a_3, \ldots$ is a sequence of integers with $0 \leq a_i \leq 9$ for all i. We define the decimal number $n.a_1a_2a_3\ldots$ to be the real number

$$\sup \left\{ n + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_k}{10^k} \middle| k \geq 0 \right\}.$$

Show that the set $\{n + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_k}{10^k} | k \geq 0\}$ is bounded above and so the sup does exist in $\mathbb{R}$.

(b). Conversely, given a real number x, following Section 1.22, let n be the largest integer such that $n \leq x$, and define the sequence $a_1, a_2, \ldots,$ inductively, where a_k is defined to be the largest integer such that $n + \frac{a_1}{10} + \cdots + \frac{a_k}{10^k} \leq x$. Show that $0 \leq a_k \leq 9$ for all $k \geq 1$. Show also that there does not exist $m > 0$ such that $a_k = 9$ for all $k \geq m$. (In other words, the natural representation of a number as a decimal does not end with an infinite sequence of nines.) Finally, show that $x = n.a_1a_2a_3\ldots$ (as defined in part (a)).

(c). Suppose that $n.a_1a_2\ldots$ and $p.b_1b_2\ldots$ are two decimal numbers, where there does not exist $m > 0$ such that $a_k = 9$ for all $k \geq m$ or such that $b_k = 9$ for all $k \geq m$. Show that if $n.a_1a_2\cdots = p.b_1b_2\ldots,$ then $n = p$ and $a_k = b_k$ for all $k \geq 1$. Conclude that each real number x has a unique decimal representation $n.a_1a_2\ldots$ not ending in an infinite sequence of nines.

Optional problem (handing in is not required)

B. Chapter 1: #20