Math 140a Fall 2015 Homework 1

Due Friday October 2 by 5pm in HW box in basement of AP&M

Reading

All references are to Rudin, 3rd edition.

Read Chapter 1.

Assigned problems (write up full solutions and hand in):

Chapter 1: #3, 4, 5, 6, 8, 9

Hints on #6: For Part (c) of #6, it may help to note that if a subset E of $\mathbb{R}$ has a maximum, that is, $x \in E$ and $y \leq x$ for all $y \in E$, then $\sup E = x$.

Part (d) of #6 is quite challenging and so here is a hint for one possible way you can approach it. There are still lots of gaps in the following proof outline which will be a challenge for you to fill in.

First go back to (c) and prove the stronger result that if $r \in \mathbb{Q}$ and

$$B'(r) = \{b^t \,|\, t \in \mathbb{Q}, t < r\},$$

then it is still true that $\sup B'(r) = b^r$. For this, you can use the following version of problem 7(e): If $w \in \mathbb{Q}$ and $b^w > y$, then $b^{w-(1/n)} > y$ if n is an integer chosen large enough. To do this, you have to prove enough of problem 7 to prove part 7(e) in the case w is rational, and notice that in this case the proof of 7(e) does not depend on problem 6(d). (7(e) for a general real w uses the result of problem 6(d) you are trying to prove).

Now show that this implies that for all real x one has $b^x = \sup B'(x)$, where $B'(x) = \{b^t | t \in \mathbb{Q}, t < x\}$. Now for $x, y \in \mathbb{R}$, show $B'(x+y) = \{b^s b^t | s, t \in \mathbb{Q}, s < x, t < y\}$. Finally, prove that $\sup\{b^s b^t | s, t \in \mathbb{Q}, s < x, t < y\} = b^x b^y$.

(Note: since $B'(x) = B(x)$ when x is irrational, the paragraph above works to show that $b^x b^y = b^{x+y}$ when x, y and $x + y$ are all irrational, without needing anything from problem 7. Perhaps you can see a way to use this to derive the case where possibly one of $x, y, x + y$ is rational without using problem 7 (I didn't).)

Problem not from the text (also to be handed in):

A. Suppose S is an ordered set with order $<$, and T is an ordered set with order $\prec$. A *isomorphism of ordered sets* is a one-to-one and onto function [1] $f : S \to T$ with the property that if $x < y$ in S, then $f(x) \prec f(y)$ in T. Two ordered sets are *isomorphic* if there exists an isomorphism of ordered sets from one to the other.

Consider the four sets

1. $\mathbb{N} = \{0, 1, 2, \dots\}$,
2. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$,
3. $\mathbb{Q} = \{a/b \,|\, a, b \in \mathbb{Z}, b \neq 0\}$, and
4. The real numbers $\mathbb{R}$.

Consider each one of these sets as an ordered set using the usual order on that number system.

Now show that no two of these ordered sets are isomorphic to each other.

Optional problem (handing in is not required)

Chapter 1: #7

[1] Most of you probably had Math 109 using Eccles' textbook, which uses the terms "injective" and "surjective". Our textbook uses the respective terms "one-to-one" and "onto" for these properties of functions. So a one-to-one and onto function is the same as a *bijective* function in the modern terminology, that is, an injective and surjective function.