Math 109 Winter 2010 Homework 10

Due 3/12/10 in class

(All exercise and page numbers refer to Eccles.)

Reading

Chapters 14, 12.

Assigned problems from the text (write up and hand in.)

In the Exercises III which begin on page 182, do

#14, 20.

Additional problems not from the text (write up and hand in.)

1. Let A_1, A_2, A_3, $\ldots$ be an infinite sequence of sets, where each set A_n is countable.

Prove that the union of all of these sets $A = \bigcup_{n=1}^{\infty} A_n$ is again countable.

Example: As one example to help your intuition, we could have $A_n = \{\frac{d}{n} | d \in \mathbb{Z}\}$, the set of all rational numbers which can be written as a fraction with denominator n, which it is easy to see is denumerable because $\mathbb{Z}$ is denumerable. Then the infinite union $\bigcup_{n=1}^{\infty} A_n$ would be equal to all of $\mathbb{Q}$. So this exercise gives another proof that $\mathbb{Q}$ is countable.

Hint for the exercise: Since each A_n is countable, we can always choose a surjective function $f_n : \mathbb{N} \to A_n$. (If A_n is denumerable, we can even choose f_n *bijective*, but if A_n is finite certainly we can still choose some surjection from $\mathbb{N}$ to A_n.) Write $a_{n,i} = f_n(i)$, so that $A_n = \{a_{n,1}, a_{n,2}, \ldots, \}$ is a listing of the elements of A_n (maybe with repeats). Then the union of this infinite collection of sets is $A = \{a_{n,i} | n \in \mathbb{N}, i \in \mathbb{N}\}$. Now how can you systematically list this entire set of elements in an sequence (maybe with repeats)? (You've seen this before).

2. This problem considers certain collections of subsets of the natural numbers $\mathbb{N}$.

(a). Let S be the collection of all *finite* subsets of $\mathbb{N}$. Prove that S is countable.

(b). Let P be the collection of *all* subsets of $\mathbb{N}$ (in other words, P is the power set of the set $\mathbb{N}$.) Prove that P is uncountable.

Hint: in class we proved that the set of all functions $f : \mathbb{N} \to \{0, 1\}$ (which we can also think of as the set of all infinite sequences of 0's and 1's) is uncountable. Try to relate this set to the set P.

(c). Let T be the collection of all *infinite* subsets of $\mathbb{N}$. Prove that T is uncountable.

3. This problem is just an expanded version of Problems III #28. A *polynomial with rational coefficients* is a function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_i \in \mathbb{Q}$ for all i. A *root* of f is a value $b \in \mathbb{R}$ for which $f(b) = 0$. A number $b \in \mathbb{R}$ is called *algebraic* if it is a root of some polynomial function with *rational* coefficients; otherwise b is called *transcendental.* For example, $\sqrt{2}/2$ is algebraic since it is a root of the polynomial function $f(x) = x^2 - (1/2)$. The numbers π and e are known to be transcendental (though this is not obvious.)

A well known theorem in algebra called the factor theorem implies that a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has at most n distinct roots in $\mathbb{R}$ (but might have fewer.) Assume this theorem.

(a). Prove that the set of all polynomial functions with rational coefficients is countable.

(a). Prove that the set of all algebraic real numbers is countable.

(b). Prove that the set of transcendental real numbers is uncountable.

Additional problem not to be handed in.

4. (This exercise is not to be handed in, but it is not optional. You should work through this question as preparation for the final.)

Go over the proofs in class and in the text that $\mathbb{R}$ is uncountable. We did a somewhat different proof of this than the text; but the main idea in any proof is the use of Cantor's diagonal trick.

Study whichever of these proofs you like best until you are sure you understand it. Then close your notes and write out a full proof that $\mathbb{R}$ is uncountable in your own words.