MATH 109 WINTER 2015 MIDTERM 2 SAMPLE SOLUTIONS

1 (10 pts). For each part, decide if the statement is true or false, and prove that your answer is correct.

(a) (5 pts). $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y < 0$.

(b) (5 pts). $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y < 0$.

Solution. (a). The statement is true. Given any $x \in \mathbb{R}$, choose $y = -x - 1$. Then $x + y = x - x - 1 = -1 < 0$.

(b). The statement is false. Suppose such an x exists. Then taking $y = -x + 1$, we must have $x + y < 0$. But $x + y = x + (-x + 1) = 1 > 0$, a contradiction.

2 (10 pts). Let $\mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}$ be the set of all nonzero real numbers. Consider the function $f : \mathbb{R}^{\times} \to \mathbb{R}^{\times}$ defined by $f(x) = 2/x$.

Prove that f is bijective. Find a formula for its inverse function f^{-1} and justify that your formula is correct.

Solution. We need to show that f is both injective and surjective. Suppose that $x_1, x_2 \in \mathbb{R}^{\times}$ and that $f(x_1) = f(x_2)$. Then $2/x_1 = 2/x_2$. Multiplying both sides by $x_1 x_2$ we get $2x_2 = 2x_1$. Dividing both sides by 2 we get $x_1 = x_2$. Hence f is injective.

Suppose that $y \in \mathbb{R}^{\times}$. Note that $2/y$ is a well-defined nonzero real number, because by definition $y \neq 0$. Thus $x = 2/y \in \mathbb{R}^{\times}$. Clearly $f(x) = f(2/y) = 2/(2/y) = y$. Thus f is surjective.

We claim that $f^{-1} : \mathbb{R}^{\times} \to \mathbb{R}^{\times}$ is given by the formula $f^{-1}(y) = 2/y$. By definition, $f^{-1}(y)$ is defined to be the unique x such that $f(x) = y$. We already showed that $f(2/y) = y$, so clearly $f^{-1}(y) = 2/y$.

3 (10 pts). Let $a_n a_{n-1} \dots a_1 a_0$ be the decimal expansion of an integer m, where $0 \leq a_i \leq 9$ for each i. Explicitly,

$$m = a_0 + (10)a_1 + (10^2)a_2 + \cdots + (10^n)a_n.$$

Prove that m is a multiple of 11 if and only if the alternating sum of the digits of m, that is $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$, is a multiple of 11.

Solution. We work with congruences modulo 11. First note that $10 \equiv (-1) (\bmod 11)$. We proved in class (also in the text) that congruence respects addition and multiplication. Thus

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in any expression involving sums and products, we can replace any element by an element which is congruent to it modulo 11, and the resulting expression will be congruent to the original expression modulo 11. Thus

$$m = a_0 + (10)a_1 + (10^2)a_2 + \cdots + (10^n)a_n \equiv a_0 + (-1)a_1 + (-1)^2 a_2 + \cdots + (-1)^n a_n \pmod{11}.$$

Now we note that for any integer b, $11|b$ if and only if $b \equiv 0 (\bmod 11)$. Thus $11|m$ if and only if $m \equiv 0(\bmod 11)$, if and only if $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n \equiv 0(\bmod 11)$ by the congruence above, if and only if $11|(a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n)$.

4 (10 pts). Let $S = \{(x, y) \in \mathbb{Z}\times\mathbb{Z} \,|\, 7x+5y = 0\}$ and $T = \{(x, y) \in \mathbb{Z}\times\mathbb{Z} \,|\, 7x+5y = 18\}$.

(a) (5 pts). Prove directly, without quoting any theorems on linear diophantine equations, that $T = \{(4 + x, -2 + y) \,|\, (x, y) \in S\}$.

(b) (5 pts). Prove directly that $S = \{(5z, -7z) \,|\, z \in \mathbb{Z}\}$, using only the following lemma:

Lemma 0.1. *Let a, b, c be integers such that* $\gcd(a, b) = 1$. *If $a|(bc)$, then $a|c$.*

Solution. (a). We need to show that the sets $T = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \,|\, 7x + 5y = 18\}$ and $T' = \{(4 + x, -2 + y) \,|\, (x, y) \in S\}$ are equal. Consider an ordered pair $(x', y') \in T'$, where $x' = 4 + x, y' = -2 + y$ for some $(x, y) \in S$. Then

$$7x' + 5y' = (7(4 + x) + 5(-2 + y) = (28 - 10) + 7x + 5y = 18 + 0 = 18,$$

since $7x + 5y = 0$ given that $(x, y) \in S$. Thus $(x', y') \in T$. This shows $T' \subseteq T$. Conversely, suppose that $(x, y) \in T$, so $7x + 5y = 18$. Then letting $x' = x - 4$, $y' = y + 2$, we have $7x' + 5y' = 7(x - 4) + 5(y + 2) = -18 + 7x + 5y = -18 + 18 = 0$, so $(x', y') \in S$. Thus $(x, y) = (4 + x', -2 + y') \in T'$. So $T \subseteq T'$. Thus $T = T'$ as required.

(b). If $(x, y) \in S$ then we have $7x + 5y = 0$ or $7x = -5y$. This equation shows that $5|(7x)$, where x is an integer. Note that $\gcd(5, 7) = 1$. Then by the lemma, $5|x$. Thus we can write $x = 5z$ for some $z \in \mathbb{Z}$. Now substituting for x in the equation $7x = -5y$ we get $35z = -5y$. Dividing both sides by 5 gives $7z = -y$. Thus $(x, y) = (5z, -7z)$ for some $z \in \mathbb{Z}$.

Conversely, any ordered pair of the form $(5z, -7z)$ is obviously in S since $7(5z)+5(-7z) = 35z - 35z = 0$.