MATH 109 WINTER 2015 MIDTERM 1 SOLUTIONS

Remember that there is not just one correct way to do a proof, and so your argument may differ from these sample solutions.

1 (10 pts). Let $Q(a, b, c)$ be the following, where a, b, c are integers:

$$\text{If } a|b \text{ and } b|c, \text{ then } a|(b + c).$$

(a) (5 pts). Prove that $Q(a, b, c)$ is true for all integers $a, b, c \in \mathbb{Z}$.

(b) (3 pts). Write down the *converse* of $Q(a, b, c)$. Is the converse true for all integers a, b, c? Justify your answer.

(c) (2 pts). Write down the *contrapositive* of $Q(a, b, c)$. Is the contrapositive true for all integers a, b, c? Justify your answer.

Solution.

(a). Let a, b, c be integers such that $a|b$ and $b|c$. By definition, this means there is an integer m such that $b = ma$ and an integer n such that $c = nb$. Then $c = nb = n(ma) = mna$. So $b + c = ma + mna = a(m + mn)$. Since m and n are integers, $m + mn$ is an integer as well. By the definition of divides, $a|(b + c)$.

(b). The converse of "If P then Q" is the proposition "If Q then P". Thus the converse of $Q(a, b, c)$ is "If $a|(b + c)$ then $a|b$ and $b|c$". This statement is not true for all integers a, b, c; for example, if $a = 2, b = 1, c = 1$, then $b + c = 2$ and so $a|(b + c)$, but $a|b$ does not hold since $2 \nmid 1$.

(c). The contrapositive of "If P then Q" is the proposition "If not Q then not P". Thus the contrapositive of $Q(a, b, c)$ is "If $a \nmid (b+c)$, then it is not true that $a|b$ and $b|c$". We could go further and use that "not (R and S)" is the same as "(not R) or (not S)" to write this as "If $a \nmid (b + c)$, then $a \nmid b$ or $b \nmid c$." We saw in class (also in the text) that the contrapositive of an implication is logically equivalent to it. Since $Q(a, b, c)$ is true for all integers a, b, c, so is the contrapositive.

2 (10 pts). Let A, B and C be sets.

(a) (5 pts). Show that $(A \cup C) - B \subseteq (A - B) \cup C$.

(b) (5 pts). Does $(A \cup C) - B = (A - B) \cup C$ for all sets A, B, C? Justify your answer.

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Solution.

(a). Suppose that $x \in (A \cup C) - B$. Thus $x \in A$ or $x \in C$, and $x \notin B$. If $x \in A$, then since $x \notin B$ we have $x \in A - B$. The other case is that $x \in C$. Thus in either case we must have $x \in A - B$ or $x \in C$, which means that $x \in (A - B) \cup C$ by definition. This shows that $(A \cup C) - B \subseteq (A - B) \cup C$.

(b). The statement is not true for all sets A, B and C. A counterexample is given by taking $A = \emptyset$, $B = \{1\}$, and $C = \{1\}$. Then $(A \cup C) - B = \{1\} - \{1\} = \emptyset$, while $(A - B) \cup C = \emptyset \cup \{1\} = \{1\}$. (In fact, any three sets such that $B \cap C$ is nonempty will suffice to give a counterexample.)

3 (10 pts). Prove that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for all integers $n \geq 2$.

Solution. We prove this by induction. The base case is $n = 2$, for which we get

$$\left(1 - \frac{1}{2^2}\right) = \frac{3}{4} = \frac{2+1}{(2)(2)}$$

so that the statement holds for $n = 2$. Now assume that

$$\prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}$$

For some $k \geq 2$. If we multiply both sides by $(1 - \frac{1}{(k+1)^2})$ we obtain

$$\left[\prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right)\right]\left(1 - \frac{1}{(k+1)^2}\right) = \prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \left(\frac{k+1}{2k}\right)\left(1 - \frac{1}{(k+1)^2}\right).$$

Some algebraic manipulation (find a common denominator) shows that the right hand side of this equation is equal to $\frac{k+2}{2(k+1)}$, proving the induction step. Thus the result holds for all $n \geq 2$ by induction.

4 (10 pts). Let $S = \{x \in \mathbb{R} | x > 0 \text{ and } x^2 > 4\}$.

(a) (5 pts). Show that $S = \{x \in \mathbb{R} | x > 2\}$.

(b) (5 pts). Show that S does not have a least element. In other words, there does not exist a number $a \in S$ such that $a \leq b$ for all $b \in S$.

Solution.

(a). Let $T = \{x \in \mathbb{R} | x > 2\}$. Suppose first that $x \in T$, so $x > 2$. Note that $x > 0$, so multiplying both sides by x gives $x^2 > 2x$. Multiplying both sides of $x > 2$ by 2 gives

$2x > 4$. By the transitive property of inequality, we get $x^2 > 4$. Thus $x \in S$. This proves that $T \subseteq S$.

On the other hand, suppose that $x \in S$, so $x > 0$ and $x^2 > 4$. Suppose that $x \leq 2$. Similarly as above, since $x > 0$ we can multiply both sides by x to get $x^2 \leq 2x$, and multiply both sides by 2 to get $2x \leq 4$; together these imply $x^2 \leq 4$. This contradicts $x^2 > 4$. Thus $x > 2$ and so $x \in T$. This proves that $S \subseteq T$. Thus $S = T$.

(b). We use the description of S proved in part (a): $S = \{x \in \mathbb{R} | x > 2\}$. Suppose that a is the least element of S. In paticular, $a \in S$ and so $a > 2$. Now let $b = (2 + a)/2$, the average of 2 and a. Clearly $2 < b < a$ since b is the average. (Here is a proof: Since $a > 2$, we have $a + 2 > 4$ and so multiplying by $1/2$ we get $(2 + a)/2 > 2$; similarly, since $a > 2$, $2a > 2 + a$ and so multiplying by $1/2$ we get $a > (2 + a)/2$.) Now since $b > 2$, $b \in S$, but $b < a$ and so a is not the least element of S, a contradiction. Thus S has no least element.