MATH 109 WINTER 2015 FINAL EXAM

Instructions: There are 85 points total. Justify all of your answers, and show all of your work in your blue book. You may use the result of one part of a problem in the proof of a later part, even if you do not complete the proof of the earlier part. You may quote basic theorems proved in the textbook or in class, unless the problem says otherwise, but do not quote the results of homework exercises.

1 (15 pts). (short answers)

(a) (4 pts). Are the following two propositional statements logically equivalent? Justify your answer with a truth table.

(i) if (not P), then Q

(ii) P or Q

(b) (5 pts). What is the remainder when 3^{1000} is divided by 10? Briefly ustify your answer using congruence modulo 10.

(c) (3 pts). Let X, Y, Z be subsets of a set U. Write down the inclusion/exclusion formula which calculates the size of $|X \cup Y \cup Z|$.

(d) (3 pts). State the pigeonhole principle.

2 (10 pts).

(a) (3 pts). State the division theorem.

(b) (7 pts). Prove that for all integers $n \in \mathbb{Z}$, $3|n$ if and only if $3|n^2$.

3 (10 pts). Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

4 (10 pts).

(a) (5 pts). Find all solutions to the equation $15x + 21y = 12$ with $x, y \in \mathbb{Z}$. Justify your answer in a few sentences by quoting the main theorems on the solution to linear diophantine equations.

(b) (5 pts). Find all solutions to the conguence $21y \equiv 12 \pmod{15}$ with $y \in \mathbb{Z}$, by using your answer to part (a). Give a few sentences justification explaining why your answer follows from part (a).

Date: March 16, 2015.

5 (10 pts). Let $f : X \to Y$ and $g : Y \to Z$ be functions. Suppose that the composition $g \circ f : X \to Z$ is a surjective function.

(a) (5 pts). Must f be surjective? Either prove it must be or give a counterexample.

(b) (5 pts). Must g be surjective? Either prove it must be or give a counterexample.

6 (10 pts).

(a) (5 pts). Let $X = \mathbb{R}^2 = \{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}\}$ be the set of all ordered pairs of real numbers, in other words the Cartesian plane. Define a relation on X by $(a, b) \sim (c, d)$ if and only if $b - d = a^2 - c^2$. Prove that $\sim$ is an equivalence relation on A.

(b) (5 pts). Describe geometrically what the equivalence classes of $\sim$ and the corresponding partition of $\mathbb{R}^2$ look like. Sketch some graphs as part of your answer.

7 (10 pts). Let a and b be positive integers. Recall that the *least common multiple* of a and b is the unique positive integer m such that (i) $a|m$ and $b|m$, and (ii) given any integer n with $a|n$ and $b|n$, then $m \leq n$.

Let m be the least common multiple of a and b. Prove that if n is any positive integer such that $a|n$ and $b|n$, then $m|n$.

8 (10 pts). Let X and Y be sets and let $f : X \to Y$ be a function. Recall that for any subset $A \subseteq X$, the *image of A under f* is defined to be $f(A) = \{y \in Y | y = f(a) \text{ for some } a \in A\}$. (Your textbook uses the notation $\vec{f}(A)$ for $f(A)$.)

(a) (5 pts). Prove that for any subsets $A_1 \subseteq X$, $A_2 \subseteq X$,

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

(b) (5 pts). With the same setup as in part (a), prove that if f is an injective function, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.