MATH 109 FALL 2016 MIDTERM 2

Instructions: Justify all of your answers, and show all. You may use the result of one part of a problem in the proof of a later part, even if you do not complete the proof of the earlier part. You may quote basic theorems proved in the textbook or in class, unless the problem says otherwise, or if reproving the result of the theorem is the point of the problem. Do not quote the results of homework exercises without reproving them.

1 (10 pts).

Recall that a *sequence* with values in a set S is a function $f : \mathbb{N} \to S$. You can think of a sequence f as an infinite ordered list $f(1), f(2), f(3), \dots$ where the elements in the list come from S.

Consider the set X of all sequences with values in $\{0, 1\}$. Show that X is uncountable.

2 (10 pts). Let $f : A \to B$ and $g : B \to C$ be functions, and consider the composition $h = g \circ f : A \to C$.

(a). Suppose that h is bijective. Show that f is injective and g is surjective.

(b). Give an example where h is bijective but g is not injective.

3 (10 pts).

(a). Carefully state the pigeonhole principle.

(b). Consider a set X consisting of 10 distinct numbers chosen from the set $\{1, 2, 3, \dots, 40\}$ of the first 40 natural numbers. Show that X must have contain two subsets Y and Z with $Y \neq Z$ and $|Y| = |Z| = 3$, such that the sum of the elements in Y is the same as the sum of the elements in Z.

4 (10 pts).

Recall that given a function $f : X \to Y$, for any subset $B \subseteq Y$ we define the inverse image of B to be the subset of X given by

$$\overleftarrow{f}(B) = \{x \in X | f(x) \in B\}.$$

(This is the book's notation; the more standard notation for this set is $f^{-1}(B)$).

(a). Show that if $B_1 \subseteq B_2 \subseteq Y$, then $\overleftarrow{f}(B_1) \subseteq \overleftarrow{f}(B_2)$.

(b). Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Show that the converse of part (a) does not hold for the function f.