MATH 109 FALL 2016 MIDTERM 2 SAMPLE SOLUTIONS

1 (10 pts).

Recall that a *sequence* with values in a set S is a function $f : \mathbb{N} \to S$. You can think of a sequence f as an infinite ordered list $f(1), f(2), f(3), \ldots$ where the elements in the list come from S.

Consider the set X of all sequences with values in $\{0, 1\}$. Show that X is uncountable.

Solution. It is easy to see that X is not finite; for example, for each integer $n \geq 1$ there is a sequence g such that $g(m) = 1$ if $m = n$ and $g(m) = 0$ if $m \neq n$. These sequences are all different for distinct n, so there must be infinitely many elements of X. Since a countable set is either finite or denumerable, to show that X is uncountable it remains to show that X is not denumerable.

Suppose for contradiction that X is denumerable. Then there is a bijection $h : \mathbb{N} \to X$, where $\mathbb{N}$ is the natural numbers. We write the sequence $h(n)$ as h_n, and think of the elements of X as the ordered list $h_1, h_2, h_3, \ldots$ where each element of X appears in the list exactly once.

Write $h_n(m) = a_{n,m}$ where $a_{n,m}$ is 0 or 1. In other words, the nth sequence h_n can be identified with the ordered list $a_{n,1}, a_{n,2}, a_{n,3}, \ldots$, and we can visualize the information we have as the following infinite array:

$$\begin{array}{lllll} h_1 : & a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ h_2 : & a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ h_3 : & a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We now construct a new sequence $f : \mathbb{N} \to \{0, 1\}$ by Cantor's diagonal argument, by changing the values of the diagonal elements of the array. Namely, we put $f(n) = \begin{cases} 1 & \text{if } a_{n,n} = 0 \\ 0 & \text{if } a_{n,n} = 1 \end{cases}$.

Suppose that $f = h_n$ for some n. Then $f(n) = h_n(n) = a_{n,n}$, but this contradicts the definition of f, which clearly has $f(n) \neq a_{n,n}$. Thus f is not equal to any of the h_n. But f is a sequence with values in $\{0, 1\}$ and thus belongs to X. Since h is a bijection, in particular it is surjective and so f should be equal to one of the h_n. This is a contradiction. Thus X is not denumerable and so is uncountable as claimed.

2 (10 pts). Let $f : A \to B$ and $g : B \to C$ be functions, and consider the composition $h = g \circ f : A \to C$.

(a). Suppose that h is bijective. Show that f is injective and g is surjective.

(b). Give an example where h is bijective but g is not injective.

Solution.

(a). let $h = g \circ f$ be bijective, so by definition it is injective and surjective.

Suppose that $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$. Then $h(a_1) = g(f(a_1)) = g(f(a_2)) = h(a_2)$. Since h is injective, $a_1 = a_2$. This proves that f is injective.

Given $c \in C$, since h is surjective, there exists $a \in A$ such that $h(a) = c$. This means $g(f(a)) = c$. Setting $b = f(a) \in B$, we then have $g(b) = c$. This shows that for all $c \in C$, there exists $b \in B$ such that $g(b) = c$. Thus g is surjective by definition.

(b). Let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1\}$. Define $f : A \to B$ by $f(1) = 1$ and $g : B \to C$ by $g(1) = 1$, $g(2) = 1$. Clearly g is not injective since $g(1) = g(2)$ although $1 \neq 2$. However, $h : A \to C$ is given by $h(1) = g(f(1)) = 1$ and since A and C each have one element, h is bijective.

3 (10 pts).

(a). Carefully state the pigeonhole principle.

(b). Consider a set X consisting of 10 distinct numbers chosen from the set $\{1, 2, 3, \ldots, 40\}$ of the first 40 natural numbers. Show that X must have contain two subsets Y and Z with $Y \neq Z$ and $|Y| = |Z| = 3$, such that the sum of the elements in Y is the same as the sum of the elements in Z.

Solution.

(a). There are several variant ways of stating the pigeonhole principle which are acceptable as an answer here. For example:

Given finite sets X and Y with $|X| > |Y|$, any function $f : X \to Y$ is not injective, that is, there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

(b). We first count the number of different 3-element subsets of X. Since X has 10 elements, this is given by the binomial coefficient $\binom{10}{3}$. We calculate this number as $10!/(3!)(7!) = (10)(9)(8)/(3)(2)(1) = 120$.

Notice that given three distinct numbers from 1 to 40, the smallest their sum can possibly be is $1 + 2 + 3 = 6$, and the largest their sum can be is $38 + 39 + 40 = 117$. Thus we can define a function

$$f : \{\text{3-element subsets } Z = \{z_1, z_2, z_3\} \subseteq X\} \to \{6, 7, \ldots, 116, 117\}$$

where f takes the subset $\{z_1, z_2, z_3\}$ to $z_1 + z_2 + z_3$, the sum of the three elements in the subset. The codomain of this function has $117 - 6 + 1 = 112$ elements, while by the calculation above the domain has 120 elements. By the pigeonhole principle, there must exist two different 3-element subsets Y and Z of X such that $f(Y) = f(Z)$, which means that the sum of the elements in Y is the same as the sum of the elements in Z.

4 (10 pts).

Recall that given a function $f : X \to Y$, for any subset $B \subseteq Y$ we define the inverse image of B to be the subset of X given by

$$\overleftarrow{f}(B) = \{x \in X | f(x) \in B\}.$$

(This is the book's notation; the more standard notation for this set is $f^{-1}(B)$).

(a). Show that if $B_1 \subseteq B_2 \subseteq Y$, then $\overleftarrow{f}(B_1) \subseteq \overleftarrow{f}(B_2)$.

(b). Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Show that the converse of part (a) does not hold for the function f.

Solution.

(a). Let $x \in \overleftarrow{f}(B_1)$. Then by definition $f(x) \in B_1$. Since $B_1 \subseteq B_2$, $f(x) \in B_2$ as well. By definition again, this means that $x \in \overleftarrow{f}(B_2)$. This shows that $\overleftarrow{f}(B_1) \subseteq \overleftarrow{f}(B_2)$.

(b). Let $B_1 = \{-1, 0\}$ and $B_2 = \{0\}$. We rely on the following theorem we have proved earlier: if $x \in \mathbb{R}$, then $x^2 \geq 0$. Also, we proved that if $xy = 0$ then either $x = 0$ or $y = 0$; thus $x^2 = 0$ implies $x = 0$.

Thus no real number has negative square, and the only number whose square is 0 is 0, so we see that $\overleftarrow{f}(B_1) = \{x \in \mathbb{R} | f(x) = x^2 \in \{-1, 0\}\} = \{0\}$ and similarly $\overleftarrow{f}(B_2) = \{x \in \mathbb{R} | f(x) = x^2 \in \{0\}\} = \{0\}$. Thus in fact $\overleftarrow{f}(B_1) = \overleftarrow{f}(B_2)$, so certainly $\overleftarrow{f}(B_1) \subseteq \overleftarrow{f}(B_2)$.

However, clearly B_1 is not a subset of B_2, since $-1 \in B_1$ while $-1 \notin B_2$. Thus the converse of (a) does not hold for f in general.

(remark: in fact the converse of (a) holds for a function f if and only the function f is surjective.)