MATH 109 FALL 2016 MIDTERM 1 – SAMPLE SOLUTIONS

Remember that there is more than one way to do a proof. Your answer may be correct even though it differs from the one here.

1 (5 pts). A *tautology* is a statement involving propositional variables $P, Q, \ldots$ which is always true no matter what propositions are substituted for the variables.

Is the statement

$$Q \Rightarrow (P \Rightarrow Q)$$

a tautology? Justify your answer.

Solution. We show the given statement is a tautology using a truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow (P \Rightarrow Q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

(Alternatively, rather than using a truth table, it is also possible to argue this through in words, as follows. Let $R = (P \Rightarrow Q)$. For $Q \Rightarrow R$ to be false, it must be that Q is true and R is false, since this is the only case where an implication is false. But if Q is true, then $P \Rightarrow Q$ is also true (again, the only way it could be false is for P to be true and Q to be false), so if Q is true than R cannot be false. Thus $Q \Rightarrow (P \Rightarrow Q)$ is always true.)

2 (5 pts). Prove the following statement.

It is not true that for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $xy = 1$.

Solution.

We are asked to prove

$$\text{Not}\, \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1.$$

By the rules for interchanging Not with quantifiers, this is logically equivalent to

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \neq 1.$$

To prove this, take $x = 0$. Then for all $y \in \mathbb{R}$, we have $xy = 0(y) = 0 \neq 1$.

Date: October 21, 2016.

3 (10 pts). In this problem you may use the following fact: an integer n is odd if and only if $n = 2m + 1$ for some integer m.

(a) Let m be an integer. Prove that $m^2 + m$ is even.

(b) Let n be an integer. Prove that n is odd if and only if $8|(n^2 - 1)$.

Solution.

(a). The integer m is either even or odd. Suppose first that m is even, so that $m = 2k$ for some integer k. Then

$$m^2 + m = (2k)^2 + 2k = 4k^2 + 2k.$$

We have $4k^2 + 2k = 2(2k^2 + k)$, where $2k^2 + k$ is an integer. thus $m^2 + m$ is even by definition.

Otherwise m is odd. We can assume in this case that $m = 2k+1$ for some integer k. Then

$$m^2 + m = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2.$$

We have $4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$, where $2k^2 + 3k + 1$ is an integer. Thus $m^2 + m$ is even in this case as well.

(b). Suppose first that n is an odd integer. We are allowed to assume that $n = 2m + 1$ for some integer m in this case. Then

$$n^2 - 1 = (2m + 1)^2 - 1 = 4m^2 + 4m + 1 - 1 = 4m^2 + 4m = 4(m^2 + m).$$

By part (a), $m^2 + m$ is always even, so $m^2 + m = 2k$ for some integer k. Then $n^2 - 1 = 4(m^2 + m) = 8k$. Thus $n^2 - 1$ is a multiple of 8, in other words $8|(n^2 - 1)$.

Conversely, suppose that $8|(n^2 - 1)$; we need to show that n is odd. We have $n^2 - 1 = 8k$ for some integer k, so $n^2 = 8k + 1 = 2(4k) + 1$. By the characterization of odd integers we are allowed to assume, this implies that n^2 is odd. Suppose for contradiction that n is even. Then $n = 2m$ for some integer m, so $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$ is also even. This is a contradiction since we know that n^2 is odd. Thus n is odd as required.

4 (10 pts). Let A, B, C be sets.

(a) Show that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

(b) Show that $A \cap (B \cup C) = (A \cap B) \cup C$ if and only if $C \subseteq A$.

Solution.

(a). Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$. Since $x \in (B \cup C)$, either $x \in B$ or $x \in C$. Suppose that $x \in B$. Then $x \in A$ and $x \in B$, so $x \in (A \cap B)$. Otherwise $x \in C$. We see that either $x \in (A \cap B)$, or $x \in C$. Thus $x \in (A \cap B) \cup C$. We have proved that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$ as required.

(b). Suppose that $C \subseteq A$. To show that $A \cap (B \cup C) = (A \cap B) \cup C$, since we have already shown one inclusion in part (a), we just need to show that $(A \cap B) \cup C) \subseteq A \cap (B \cup C)$. Suppose that $x \in (A \cap B) \cup C$. Either $x \in (A \cap B)$ or $x \in C$. If $x \in (A \cap B)$, then $x \in A$

and $x \in B$. Since $x \in B$, certainly $x \in (B \cup C)$. So $x \in A$ and $x \in (B \cup C)$, and thus $x \in A \cap (B \cup C)$. Otherwise $x \in C$. In this case, by the hypothesis $C \subseteq A$, we get also that $x \in A$. Since $x \in C$, certainly $x \in (B \cup C)$. Again we get $x \in A$ and $x \in (B \cup C)$ and so $x \in A \cap (B \cup C)$. This proves that $(A \cap B) \cup C) \subseteq A \cap (B \cup C)$.

Conversely, suppose that $A \cap (B \cup C) = (A \cap B) \cup C$. If $x \in C$, then certainly $x \in (A \cap B) \cup C$. Thus $x \in A \cap (B \cup C)$, since this set is assumed equal. But this means that $x \in A$ and $x \in (B \cup C)$, so in particular $x \in A$. Since $x \in C$ implies $x \in A$, we have $C \subseteq A$ in this case as we wished.

5 (10 pts). We define a sequence of numbers by induction as follows. Let $v_1 = 3$, $v_2 = 5$, and define $v_{n+1} = 2v_n + v_{n-1}$ for all $n \geq 2$.

(a) Calculate v_5.

(b) Prove that $2^n \leq v_n \leq 3^n$ for all $n \geq 1$.

Solution.

(a) By the inductive definition of the v_n, we have

$v_3 = 2v_2 + v_1 = 2(5) + 3 = 13$,

$v_4 = 2v_3 + v_2 = 2(13) + 5 = 31$, and finally

$v_5 = 2v_4 + v_3 = 2(31) + 13 = 75$.

(b) For clarity, we prove that $2^n \leq v_n$ for all $n \geq 1$ and that $v_n \leq 3^n$ for all $n \geq 1$ in two separate (strong) induction proofs.

First, $2 = 2^1 \leq v_1 = 3$ and $4 = 2^2 \leq v_2 = 5$, proving the base cases. For the induction step, assume that $2^m \leq v_m$ for all $1 \leq m \leq k$, some $k \geq 2$. We have $v_{k+1} = 2v_k + v_{k-1}$. By the induction hypothesis, $2^k \leq v_k$, and multiplying by 2 we get $2^{k+1} \leq 2v_k$. We also have $2^{k-1} \leq v_{k-1}$, so in particular $0 \leq v_{k-1}$. Thus $2^{k+1} \leq 2v_k \leq 2v_k + v_{k-1} = v_{k+1}$, proving the induction step. Thus $2^n \leq v_n$ for all $n \geq 1$, by induction.

Next, $v_1 = 3 \leq 3^1 = 3$ and $v_2 = 5 \leq 3^2 = 9$, so the base cases are correct. For the induction step, assume that $v_m \leq 3^m$ for all $1 \leq m \leq k$, some $k \geq 2$. We have $v_{k+1} = 2v_k + v_{k-1}$. By the induction hypothesis, $v_k \leq 3^k$ and $v_{k-1} \leq 3^{k-1}$. Thus $v_{k+1} = 2v_k + v_{k-1} \leq 2(3^k) + 3^{k-1}$. Multiplying both sides of $1 < 3$ by 3^{k-1}, we also get $3^{k-1} < 3^k$, so $2(3^k) + 3^{k-1} \leq 2(3^k) + 3^k = 3(3^k) = 3^{k+1}$. Putting these inequalities together we get $v_{k+1} \leq 3^{k+1}$, proving the induction step. Thus $v_n \leq 3^n$ for all $n \geq 1$ by induction as well.