Math 109 Fall 2016 Homework 6, due 11/4/2016 in HW boxes in the basement of AP&M by 3 pm

1 Reading and practice

Read Chapter 14, and begin to read Chapters 15-16 which we will cover next. Do the end of chapter exercises as you read, and check your work against the answers in the back. These exercises are to test your understanding and they are not to be written up and handed in.

2 Exercises to submit on Friday 10/28

2.1 Exercises from the text

In the Problems III which begin on page 182, do #14, 18, 20.

Remarks: Note that in #14, there is a typo: N_{2n} should be $\mathbb{N}_{2n}$. Recall that, as defined on page 124 of the text, this means the set $\{1, 2, \ldots, 2n\}$ of the first $2n$ natural numbers.

There is also a typo in #18: in the formula you are supposed to prove the sum should run from $i = 0$ to $i = k$, not from $i = 0$ to $i = n$. In other words, you should prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

Hint for #20: I suggest you first prove there are two subsets with the same sum, without worrying about whether or not they are disjoint. Then show how to adjust your subsets in order to produce two disjoint subsets with the same sum.

Additional problems (write up and hand in)

1. Prove that if X and Y are countable sets, then $X \cup Y$ is also countable. (remember that countable means denumerable or finite).

2. Recall that a *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$. Writing $a_i = f(i)$, we can think of the sequence given by f as an infinite ordered list of real numbers $a_1, a_2, a_3, \ldots$ (where the list can have repeats).

Let X be the set of all sequences $f : \mathbb{N} \to \mathbb{R}$ such that the image of f is contained in $\{0, 1\}$. For example, one such sequence is $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$. In other words this sequence looks like $1, 0, 1, 0, 1, 0, \ldots$.

Prove that X is uncountable. (Use a variation of Cantor's diagonal trick).

3. (a). Let Y be the set of all subsets of $\mathbb{N}$, in other words $Y = \mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. Let X be the set of all sequences with image in $\{0, 1\}$ considered in problem 2. Show that X and Y are equipotent. Conclude that Y is uncountable.

(b). Let Z be the set of all *finite* subsets of $\mathbb{N}$. Show that Z is countable.

(c). Let W be the set of all *infinite* subsets of $\mathbb{N}$. Show that W is uncountable. (Note that $Y = Z \cup W$).

4. Recall that if $a, b \in \mathbb{R}$ and $a < b$, then we define the *open interval*

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

(Caution: do not confuse this with an ordered pair. Unfortunately, the notation for open intervals in $\mathbb{R}$ is the same as the notation for ordered pairs. No ordered pairs occur in this problem.)

(a). Let a, b, c, d be four real numbers with $a < b$, and $c < d$. Prove that the sets (a, b) and (c, d) are equipotent by finding a bijective function $f : (a, b) \to (c, d)$.

(b). By considering the function $y = \tan x$, find an open interval (c, d) which is equipotent to the entire real line $\mathbb{R}$.

(c). Let a, b be any two real numbers with $a < b$. Prove that the interval (a, b) and $\mathbb{R}$ are equipotent.