Math 103b Winter 2008 Exam 2 review sheet

Review all of the homework problems on homeworks 4-5, as well as your notes. Below are some specific things I want to point out to you that you are expected to know.

•Know what the polynomial ring $R[x]$ is, for any commutative ring R with unity.

•Know when $R[x]$ is an integral domain (this is if and only if R is an integral domain) and why this is true.

•Understand the statement of the division algorithm for the ring $F[x]$ where F is a field. Be able to state the algorithm, and perform it over various fields F (This is easy for $F = \mathbb{R}$, $\mathbb{Q}$, or $\mathbb{C}$, but takes a little more thought if $F = \mathbb{Z}_p$.)

•Understand the Remainder and Factor Theorems, and their proofs. We did the proofs in class.

•Know the definition of a PID (principal ideal domain), and the theorem that $F[x]$ (for a field F) is a PID. Understand the basic idea of the proof, and the fact in Theorem 16.4 that any ideal I is equal to $\langle g \rangle$ where g can be any nonzero polynomial in I of minimal degree.

•Be able to prove facts like $\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$ (Example 3 in Chap 16), using the 1st isomorphism theorem and Theorem 16.4. The method of Chap 16 Exercise 40 is similar.

0.1 Chapter 17

•Understand the definition of reducible and irreducible polynomials in $R[x]$. Understand the differences between the cases where $R = \mathbb{Z}$ or $R = F$ is a field.

•Be able to decide if a polynomial $f \in F[x]$ of degree 2 or 3 is irreducible, by checking if f has a root in F. If $F = \mathbb{Q}$ you can use the rational-root test to see if it has a root. If $F = \mathbb{Z}_p$ you check if f has a root by trial and error.

•Understand the mod p irreducibility test. This can sometimes be used to prove that polynomials in $\mathbb{Z}[x]$ of degree 4 or higher are irreducible over $\mathbb{Q}$. See examples in book and homework exercises.

•Understand the theorem that $\langle f(x) \rangle$ is a maximal ideal of $F[x]$ if and only if $f(x)$ is irreducible. Be able to use this to decide if a (principal) ideal in $F[x]$ is maximal.

•Understand how to use this to create new fields. Given any irreducible polynomial $f(x)$ in $F[x]$, $E = F[x]/\langle f(x) \rangle$ is a field. If $F = \mathbb{Z}_p$ and f has degree n, then E will be a field with p^n elements— understand the basic idea why that is true (Exercise 17.6.) Be able to create fields with p^n elements this way, for small p and n. (Say $n = 2$ or 3; then one just needs to find an irreducible polynomial of degree n over $\mathbb{Z}_p$, which can be done by trial and error.)

0.2   Chapter 18

•Understand definitions of irreducibles, primes, and associates in any commutative domain R with unity. (The definitions of irreducibles for $F[x]$ above are just a special case.)

•Understand what elements are units and what elements are associates of each other in simple rings like $\mathbb{Z}$ and $F[x]$.

•Be able to work with the rings $\mathbb{Z}[\sqrt{d}]$. We always assume that d is not divisible by the square of a prime. Then the *norm* function $N(a + b\sqrt{d}) = |a^2 - db^2|$ has lots of useful properties which you should know: $(1) N(xy) = N(x)N(y)$; (2) $N(x) = 1$ if and only if x is a unit; (3) $N(x) = 0$ if and only if $x = 0$; (4) if $N(x)$ is a prime number then x is irreducible. *understand the proofs of facts (2-4), which we did in class. The proof of (1) is a boring calculation so I won't ask you to worry about it.*

•Understand the examples in class and the homework exercises about $\mathbb{Z}[\sqrt{d}]$. (In class we looked at certain elements in $\mathbb{Z}[\sqrt{-3}]$ and in $\mathbb{Z}[\sqrt{5}]$.) Understand the general techniques for working in these rings using the norm.

For example, be able to show certain elements in these rings are irreducible or not, prime or not, using the methods in those examples above.

•Understand the relationship between prime and irreducible elements. Prime implies irreducible; but there are rings with elements that are irreducible but not prime (for example $\mathbb{Z}[\sqrt{-3}]$). In a PID, prime and irreducible elements are the same thing.

•Know the definition of a UFD (unique factorization domain).

•Know the theorem that a PID is a UFD. Understand that this means that $\mathbb{Z}$ and $F[x]$ are UFDs. Understand how to write an element in these rings as a product of irreducibles, and what "uniqueness up to associates" means in $F[x]$.

•Know the definition of a Euclidean domain (ED). Know the theorem that a ED is a PID, as well as the general idea of the proof. The proof is just a small generalization of the proof that $F[x]$ is a PID.

•Know that $\mathbb{Z}[i]$ is a ED, and thus a UFD. We proved this in class.

•Understand the relationships between the various kinds of rings. ED implies PID, and PID implies UFD.