Math 103b Winter 2008 Exam 1 review sheet

• The exam will cover chapters 12-15 of the book. It is closed book and closed notes (and no calculators.)

Definitions you should know

• Ring (but I will not ask you to check that something is or is not a ring directly from the definition).
• Identity element of a ring.
• Commutative ring.
• Unit.
• Subring.
• Zero-divisor.
• Integral Domain. (Recall this means that R is commutative with unity, and that ab = 0 implies a= 0 or b= 0.)
• Field.
• Cancelation property.
• Characteristic of a ring. (I only care about characteristic for rings R with identity, and I define it to be the smallest positive integer n such that $n \cdot 1 = 0$, or if no such n exists the characteristic is defined to be 0.)
• Ideal.
• Factor ring.
• Prime ideal.
• Maximal ideal.
• Homomorphism and isomorphism.
• Kernel and image of a homomorphism.

Examples of Rings

We have only studied a few classes of rings. You should know all of these and their basic properties. How do you multiply and add in each one?

Which are integral domains and which aren't? Which are fields? Which are commutative and which are noncommutative? What is the characteristic of each ring? Which elements in the ring are units?

- Rings of numbers: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
- $\mathbb{Z}_m$, the integers modulo m, for any $m \geq 2$.
- Matrix rings: $M_2(F)$, which is 2×2-matrices with entries from F. Here F could be any of the rings of numbers above, or even $\mathbb{Z}_m$ for some m.
- The Gaussian integers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$.
- The ring of polynomials $F[x]$, which consists of all elements of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the coefficients a_i all come from F. Here F could be any of the rings of numbers above, or even $\mathbb{Z}_m$ for some m.
- The ring $\mathbb{Q}[\sqrt{m}]$, where m is a positive integer which is not a square. This ring consists of all elements $\{a + b\sqrt{m} | a, b, \in \mathbb{Z}\}$. (I only did the case $m = 2$ on the board, but the same construction works for any m.)
- Given any two rings R and S, the direct sum of R and S is a new ring $R \oplus S = \{(r, s) | r \in R, s \in S\}$, with component-wise addition and multiplication.

Important theorems and techniques

- Know how to check if a subset of a ring is a subring.
- Know that $\mathbb{Z}_m$ is a field precisely when m is prime, and understand why $\mathbb{Z}_m$ fails to be even an integral domain when m is not prime.
- Understand the example $\mathbb{Q}[\sqrt{2}]$ and *understand the proof that it is a field.*
- understand the theorem that the characteristic of a domain is a prime number (or 0).
- Know how to check if a subset of a ring is an ideal of the ring.
- Given a commutative ring R with element a, know the definition of the *principal ideal generated by* a, written $\langle a \rangle$.
- Understand the definition of a factor ring and how to do addition and multiplication in such a ring.
- Understand some important examples where factor rings can be shown to be the same as other familiar rings. For example:

$$\mathbb{Z}/\langle m \rangle \cong \mathbb{Z}_m. \quad \mathbb{R}[x]/\langle x \rangle \cong \mathbb{R}. \quad \mathbb{Z}[i]/\langle 2 - i \rangle \cong \mathbb{Z}_5.$$

- Know the theorem that a ideal I of a commutative ring R is prime if and only if R/I is a domain, and that I is maximal if and only if R/I is a field. As a special case, know that a commutative ring R with unity is a field if and only if R and $\{0\}$ are the only ideals of R.

• Be able to check if a function between two rings is a homomorphism, and if it is a isomorphism.

• Know that the kernel of a homomorphism $\phi : R \to S$ is always an ideal of R , and the image of a homomorphism is always a subring of S. Know the statement of the 1st isomorphism theorem: $R/\ker\phi \cong \operatorname{Im}\phi$ and how to use it.

• Know that given any ring R with identity, there is a homomorphism $\mathbb{Z} \to R$ sending a to $a \cdot 1$. The kernel of this homomorphism is exactly $\langle m \rangle$, where m is the characteristic of R.

Homework

Review the homework problems on homeworks 1-3.