Math 103b Spring 2014 Sample Midterm 2

1. (a) Define what it means for a ring R to be a PID (principal ideal domain).

(b) Define what it means for a ring R to be a UFD (unique factorization domain).

(c). Consider the following rings:

$$\mathbb{Z},\ \mathbb{Q}[x],\ \mathbb{Z}_6[x],\ \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\},\ \mathbb{Z}[x].$$

(c1). Which rings on the list above are UFD's? (no proof necessary)

(c2). Which rings on the list above are PID's? (no proof necessary)

2. Suppose that I is an ideal of the ring $\mathbb{Q}[x]$. You are given that $x^3 - 1 \in I$ and $x^2 - 1 \in I$. You are also given that I is a proper ideal (in other words $I \neq \mathbb{Q}[x]$.) What is I? Prove your answer.

3(a). Let $f(x) = x^4 + x + \overline{1} \in \mathbb{Z}_3[x]$. Is f irreducible in $\mathbb{Z}_3[x]$? Justify your answer.

(b). Let $f(x) = x^3 - x - 5 \in \mathbb{Q}[x]$. Is $\mathbb{Q}[x]/\langle f \rangle$ a field? Justify your answer.

4. This problem is about $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} | a, b \in \mathbb{Z}\}$. Recall that the *norm* of any element $x = a + b\sqrt{5}$ is $N(a + b\sqrt{5}) = |a^2 - 5b^2|$. The following are the properties of the norm we studied, which you can assume without proof.

(i) $N(x) = 0$ if and only if $x = 0$;

(ii) $N(x)N(y) = N(xy)$ for all $x, y \in R$;

(iii) x is a unit in R if and only if $N(x) = 1$; and

(iv) if $N(x)$ is a prime number in $\mathbb{Z}$, then x is irreducible in R.

(a). Prove that no element of R has norm 2. (Hint: consider the equation $a^2 - 5b^2 = \pm 2$ modulo 5.)

(b). Prove that $2, 1 + \sqrt{5}$, and $-1 + \sqrt{5}$ are irreducible in R.

(c). Note that $4 = (2)(2) = (1 + \sqrt{5})(-1 + \sqrt{5})$. Is R a UFD? Justify your answer.