Math 103A Fall 2005 Final

December 9, 2005

Show all work. Problems whose answers are a single number or yes or no may not get credit if there is no explanation for the answer. You may quote without proof any theorems or results we stated in class or that are in the book, but say what you are using. You may not quote the result of a homework exercise without proof.

NAME:

Problem 1 /20	
Problem 2 /25	
Problem 3 /15	
Problem 4 /15	
Problem 5 /15	
Problem 6 /25	
Problem 7 /15	
Problem 8 /20	
Total /150	

Problem 1 (20 points)

1a (5 pts) Write down the list of all possible abelian groups of order 90 up to isomorphism.

1b (10 pts) Suppose G is an abelian group of order 90, and that G has exactly 2 elements of order 6. Which group on the list of part (a) is G isomorphic to? Explain how you arrived at your answer.

1c (5 pts) Is the group G of part (b) cyclic? Why or why not?

Problem 2 (25 points)

2a (10 pts) Let $\phi : G \to \overline{G}$ be a homomorphism between two groups, where $|G| = 15$ and $|\overline{G}| = 16$. The 1st isomorphism theorem for homomorphisms states that $G/(\ker \phi) \cong \phi(G)$. Using this, prove that ϕ is the trivial homomorphism (i.e. show that $\phi(g) = e$ for all $g \in G$, where e is the identity element of $\overline{G}$.) If you use other theorems as well, mention which ones you are using.

2b (5 pts) How many homomorphisms $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_{15}$ are there? Explain your answer.

2c (10 pts) How many of the homomorphisms you found in part (b) have the additional property that $\phi(G) = \{0, 3, 6, 9, 12\}$? For any such homomorphism, what is $\phi(5)$? Justify your answer.

Problem 3 (15 points)

Let $G = U(20)$, the group of all integers which are relatively prime to 20, with operation multiplication modulo 20. Let $L = \{1, 11\}$.

3a (5 pts) Show that L is a normal subgroup of G.

3b (5 pts) Let G and L be as in part (a). Find the order of the element $(3)L$ in the factor group G/L.

3c (5 pts)

Prove that G/L is isomorphic to the group $\mathbb{Z}_4$. (Hint: it is possible to show this without constructing the isomorphism ϕ directly.)

Problem 4 (15 points)

Let G be a group of order 30, and let H be a subgroup of G. Suppose g_1, g_2, g_3, g_4, g_5 are five *distinct* elements of G. Suppose you know that $\{g_1, g_2, g_3\}$ is one of the left cosets of H, and that $\{g_1, g_4, g_5\}$ is one of the right cosets of H.

4a (5 pts) How many distinct left cosets of H in G are there? Explain your answer.

4b (5 pts) Is H normal in G? Explain why or why not.

4c (5 pts) Is it possible that $g_1 = e$, where e is the identity element of G? Explain why or why not.

Problem 5 (15 points)

5a (10 pts) Let $G = \langle x \rangle$ be a cyclic group of order 12. For which elements $x^m \in G$ does the cyclic subgroup $\langle x^m \rangle$ have order 4? Show your work.

5b (5 pts) Let a be an element of a group G. If $|a^5| = 25$, find $|a|$. Justify your answer.

Problem 6 (25 points)

In this problem, consider the group D_4, the dihedral group of symmetries of a square.

6a (5 pts) Explain in words why the product of two reflections in D_4 is a rotation.

6b (10 pts) By drawing a square and labeling the corners of the square with the numbers $1, 2, 3, 4$, show how each element in D_4 may be identified with a permutation in S_4. Write down the 8 corresponding permutations.

6c (10 pts) Let $R_{90} \in D_4$ be the symmetry which rotates the square 90 degrees clockwise. Show that R_{90} is not in $Z(D_4)$, the center of D_4.

Problem 7 (15 points)

Let β be the following element in S_5, given in cycle notation: $\beta = (123)(1425)$.

7a (5 pts)
Write β in disjoint cycle form.

7b (5 pts)
Find β^{99}.

7c (5 pts)

Suppose that $\alpha \in S_5$ is another permutation. Let $\gamma = \alpha\beta\alpha^{-1}$. Exactly one of the following statements is true: (i) γ even; (ii) γ is odd; (iii) γ is sometimes even and sometimes odd, depending on α. Which one is true? Explain your answer.

Problem 8 (20 points)

8a (10 pts) Is the set

$$H = \left\{ \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \middle| x, y \in \mathbb{R}, y \neq 0 \right\}$$

a subgroup of $\mathrm{GL}(2, \mathbb{R})$? Why or why not?

8b (10 pts) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$. Find the centralizer subgroup $C(A)$.