Math 103A Fall 2007 Exam 2

November 19, 2007

NAME:

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Problem 1 (25 points)

(a) (5 pts). Clearly state Lagrange's theorem.

(b) (20 pts) Let p be a prime number. Let G be a group with $|G| = p^n$ for some $n \geq 1$ (such a group is called a *p-group*.) Prove that G has at least one element of order p. (Hint: if you don't know how to start, consider first the special case where $|G| = 9$.)

Problem 2 (25 points)

(a) (10 pts) Let G and $\overline{G}$ be two groups. Define what it means for a function $\phi : G \to \overline{G}$ to be an isomorphism of groups.

(b) (15 pts) Define the following set of matrices:

$$G = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| a \in \mathbb{Z} \right\}.$$

The set G is a group under matrix *multiplication* (you can assume this.)

Prove that $G \cong \mathbb{Z}$, in other words that G is isomorphic to the group of integers with the operation of *addition*. Hint: you need to find a function ϕ which gives the isomorphism—try something simple.

Problem 3 (20 points)

In this problem, we consider the group S_7 of permutations of $\{1, 2, 3, \ldots, 7\}$.

(a) (10 pts). Write the permutation $\alpha = (156)(3547)$ in *disjoint* cycle form. What is the order of this permutation in the group S_7?

(b) (10 pts). Explain why the permutation α in part (a) is an odd permutation. Then find a permutation $\beta \in S_7$ which is an *even* permutation but which has the same order in S_7 as the element α. Again briefly explain your answer.

Problem 4 (30 points)

In this problem, consider the following four groups: A_4, $\mathbb{Z}_{12}$, $U(21)$, D_6. These groups all have order 12 (you don't have to prove this.) (Note that D_6 is the group of all symmetries of a regular hexagon so it contains six rotations and six reflections.)

(a) (10 pts) For each of the four groups, decide if it is Abelian or non-Abelian and list your answers below. *Prove* your answer only for the alternating group A_4.

(b) (20 pts) Prove that no two of the four groups A_4, $\mathbb{Z}_{12}$, $U(21)$, D_6 are isomorphic. You can assume without proof all of the basic properties of isomorphisms. (Starting hint: look for some elements of order 2 in $U(21)$.)

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