Math 103A Fall 2006 Exam 1

NAME: Answers

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Problem 1 (30 points)

1 Let $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, S_1, S_2, S_3, S_4\}$ be the dihedral group of order 8, which consists of symmetries of a square. Here, each R_i is the symmetry of the square given by *counterclockwise* rotation by i degrees. Each S_i is a reflection about an axis of symmetry of the square, labeled as follows:

(On the actual exam, pictures indicated the axis of reflection. In the notation of Chapter 1 of Gallian, $S_1 = H$, $S_2 = V$, $S_3 = D$, $S_4 = D'$.)

(a) (10 pts). Calculate the products $R_{270}S_1$ and S_1R_{270} in D_4. Show your work.

(b) (10 pts). Complete the following Cayley table of the group D_4, using part (a) and your knowledge of Cayley tables. (Remember that the product $g_1 g_2$ is written in row g_1 and column g_2 of the Cayley table.) Mention briefly (without proof) what facts about Cayley tables you relied on to complete the diagram.

	R_0	R_{90}	R_{180}	R_{270}	S_1	S_2	S_3	S_4
R_0	R_0	R_{90}	R_{180}	R_{270}	S_1	S_2	S_3	S_4
R_{90}	R_{90}	R_{180}	R_{270}	R_0	S_4	S_3	S_1	S_2
R_{180}	R_{180}	R_{270}	R_0	R_{90}				
R_{270}	R_{270}	R_0	R_{90}	R_{180}				
S_1	S_1	S_3			R_0	R_{180}	R_{90}	R_{270}
S_2	S_2	S_4			R_{180}	R_0	R_{270}	R_{90}
S_3	S_3	S_2			R_{270}	R_{90}	R_0	R_{180}
S_4	S_4	S_1			R_{90}	R_{270}	R_{180}	R_0

(c) (10 pts). Prove that the group D_4 is not cyclic.

Problem 2 (20 points)

(a) (10 pts) Give an example of an infinite cyclic group. Explain how you know it is cyclic.

(b) (10 pts) Give an example of an infinite non-Abelian group. Prove your example is non-Abelian.

Problem 3 (30 points)

(a) (10 pts). Consider the group $U(20)$. List the elements of $U(20)$. What is the order of $U(20)$?

(b) (10 pts). Find (by inspection, say) the inverse in $U(20)$ of the element [13], and show that your answer really is the inverse of [13].

(c) (10 pts). Let H be a subgroup of $U(20)$ such that $[9] \in H$ and $[11] \in H$, but $H \neq U(20)$. Find such an H, and prove that you answer is the *only* subgroup of $U(20)$ with those properties.

Problem 4 (20 points)

Let $G = (\mathbb{Q}, +)$ be the group of all rational numbers under the operation of addition. Explicitly,

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Let H be the subset of all rational numbers which can be written as a fraction where the denominator is 2 to some nonnegative integer power. In other words,

$$H = \left\{ \frac{a}{b} \;\middle|\, a, b \in \mathbb{Z}, b = 2^n \text{ for some } n \geq 0 \right\}.$$

(a) (15 pts). Prove that H is a subgroup of G. (Remember the operation is addition!)

(b) (5 pts). Consider the left cosets of H in $(\mathbb{Q}, +)$. Are the two cosets $(3/5) + H$ and $(17/20) + H$ equal or not? Justify your answer.