Math 103A Fall 2006 Exam 1 w/solutions

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Problem 1 (30 points)

1 Let $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, S_1, S_2, S_3, S_4\}$ be the dihedral group of order 8, which consists of symmetries of a square. Here, each R_i is the symmetry of the square given by *counterclockwise* rotation by i degrees. Each S_i is a reflection about an axis of symmetry of the square, labeled as follows:

(On the actual exam, pictures indicated the axis of reflection. In the notation of Chapter 1 of Gallian, $S_1 = H$, $S_2 = V$, $S_3 = D$, $S_4 = D'$.)

(a) (10 pts). Calculate the products $R_{270}S_1$ and S_1R_{270} in D_4. Show your work.

Refer to the method of calculation of products in the Dihedral group given in Chapter 1 of Gallian, and the Cayley table given there. The answers were $R_{270}S_1 = S_3$ and $S_1R_{270} = S_4$. The most common mistake was multiplying the elements in the wrong order: $R_{270}S_1$ means first do the reflection S_1, then do the rotation R_{270}.

(b) (10 pts). Complete the following Cayley table of the group D_4, using part (a) and your knowledge of Cayley tables. (Remember that the product $g_1 g_2$ is written in row g_1 and column g_2 of the Cayley table.) Mention briefly (without proof) what facts about Cayley tables you relied on to complete the diagram.

Once the products $S_1 R_{270}$ and $R_{270} S_1$ were filled in, the rest of the entries followed by pure logic using that every element of the group must appear exactly once in each column and exactly once in each row. The entries that needed to be filled in are in bold.

	R_0	R_{90}	R_{180}	R_{270}	S_1	S_2	S_3	S_4
R_0	R_0	R_{90}	R_{180}	R_{270}	S_1	S_2	S_3	S_4
R_{90}	R_{90}	R_{180}	R_{270}	R_0	S_4	S_3	S_1	S_2
R_{180}	R_{180}	R_{270}	R_0	R_{90}	$\mathbf{S_2}$	$\mathbf{S_1}$	$\mathbf{S_4}$	$\mathbf{S_3}$
R_{270}	R_{270}	R_0	R_{90}	R_{180}	$\mathbf{S_3}$	$\mathbf{S_4}$	$\mathbf{S_2}$	$\mathbf{S_1}$
S_1	S_1	S_3	$\mathbf{S_2}$	$\mathbf{S_4}$	R_0	R_{180}	R_{90}	R_{270}
S_2	S_2	S_4	$\mathbf{S_1}$	$\mathbf{S_3}$	R_{180}	R_0	R_{270}	R_{90}
S_3	S_3	S_2	$\mathbf{S_4}$	$\mathbf{S_1}$	R_{270}	R_{90}	R_0	R_{180}
S_4	S_4	S_1	$\mathbf{S_3}$	$\mathbf{S_2}$	R_{90}	R_{270}	R_{180}	R_0

(c) (10 pts). Prove that the group D_4 is not cyclic.

One method is to calculate every cyclic subgroup $\langle x \rangle$ for every $x \in D_4$ and prove that none of those cyclic subgroups is equal to D_4.

A less calculation-oriented argument (which many people noticed) goes as follows. Suppose R_i is a rotation in D_4. Then since every power of a rotation is again another rotation, the cyclic subgroup $\langle R_i \rangle$ contains no reflections and thus cannot be equal to D_4.

If instead S_j is a reflection in D_4, then S_j has order 2 and so $\langle S_j \rangle = \{R_0, S_j\}$, which is again not all of D_4.

Since any element of D_4 is either a reflection or a rotation, for no $x \in D_4$ will we have $\langle x \rangle = D_4$ and so D_4 is not cyclic.

Problem 2 (20 points)

(a) (10 pts) Give an example of an infinite cyclic group. Explain how you know it is cyclic.

The example is $(\mathbb{Z}, +)$, the group of integers under addition. It is cyclic because $\langle 1 \rangle = \mathbb{Z}$, in other words 1 is a generator. This is because $\langle 1 \rangle = \{n \cdot 1 | n \in \mathbb{Z}\}$, the set of all positive and negative *multiples* of 1 (remember in an additive group one uses multiples not powers), which is clearly equal to $\mathbb{Z}$.

(In fact $\langle -1 \rangle = \mathbb{Z}$ as well, but no other elements besides 1 and -1 generate $\mathbb{Z}$.)

(b) (10 pts) Give an example of an infinite non-Abelian group. Prove your example is non-Abelian.

The example is $\mathrm{GL}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$.

It is obviously infinite, since $\mathbb{R}$ is infinite and every matrix $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ with $a \in \mathbb{R}$ is in $\mathrm{GL}(2, \mathbb{R})$. (Although note that I did not require a proof that your example was infinite.)

To see that $\mathrm{GL}(2, \mathbb{R})$ is non-Abelian, it is sufficient to find a pair of matrices which do not commute. Almost any random pair of different matrices works, for example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, since $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. (However, note that there *are* some pairs of matrices which do commute, for instance any matrix commutes with itself.)

Problem 3 (30 points)

(a) (10 pts). Consider the group $U(20)$. List the elements of $U(20)$. What is the order of $U(20)$?

$U(20) = \{[a] \mid \gcd(a, 20) = 1\} = \{[1], [3], [7], [9], [11], [13], [17], [19]\}$. In other words, this consists of all congruence classes mod 20 which have a representative relatively prime to 20. There are 8 such distinct classes and so $|U(20)| = 8$.

(b) (10 pts). Find (by inspection, say) the inverse in $U(20)$ of the element [13], and show that your answer really is the inverse of [13].

This can be done by multiplying [13] by the other elements in turn until the product is [1]. Or, note that since one needs the final digit of the product with [13] to be 1, one can eliminate all but [7] and [17] as possibilities immediately. but $[13][7] = [91] = [11]$, so we must have $[13]^{-1} = [17]$. Indeed, $[13][17] = [-7][-3] = [21] = [1]$.

(c) (10 pts). Let H be a subgroup of $U(20)$ such that $[9] \in H$ and $[11] \in H$, but $H \neq U(20)$. Find such an H, and prove that you answer is the *only* subgroup of $U(20)$ with those properties.

Suppose that H is a subgroup of $U(20)$ which is not all of $U(20)$, and which contains [9] and [11]. Every subgroup contains the identity so also $[1] \in H$, and since H is closed under products, $[9][11] = [99] = [19] \in H$. By Lagrange's Theorem, $|H|$ must be a divisor of $|U(20)| = 8$, and since $H \neq U(20)$ we know that $|H| < 8$. But we have already showed that H must have at least 4 elements, which forces $H = \{[1], [9], [11], [19]\}$.

If we want to be very careful, we should also check that H really is a subgroup. This can be done by showing it is closed under products and inverses by direct computation.

Problem 4 (20 points)

Let $G = (\mathbb{Q}, +)$ be the group of all rational numbers under the operation of addition. Explicitly,

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Let H be the subset of all rational numbers which can be written as a fraction where the denominator is 2 to some nonnegative integer power. In other words,

$$H = \left\{ \frac{a}{b} \;\middle|\, a, b \in \mathbb{Z}, b = 2^n \text{ for some } n \geq 0 \right\}.$$

(a) (15 pts). Prove that H is a subgroup of G. (Remember the operation is addition!)

To show that H is a subgroup we use the 2-step subgroup test. Since G is a group with operation called addition, we must check that H is closed under sums, and closed under additive inverses. So let $h_1 = a_1/2^m$ and $h_2 = a_2/2^n$ be two elements of H, for some $a_1, a_2 \in \mathbb{Z}$ and $m, n \geq 0$. To add h_1 and h_2 we find a common denominator, and so get

$$h_1 + h_2 = a_1/2^m + a_2/2^n = (a_1 2^n + a_2 2^m)/(2^m 2^n) = (a_1 2^n + a_2 2^m)/2^{m+n}.$$

Since the numerator is again an integer and the denominator is a power of 2, $h_1 + h_2 \in H$. Thus H is closed under sums.

To check that H is closed under inverses, just note that $-h_1 = (-a_1)/2^m \in H$. Thus H is a subgroup by the 2-step subgroup test.

(b) (5 pts). Consider the left cosets of H in $(\mathbb{Q}, +)$. Are the two cosets $(3/5) + H$ and $(17/20) + H$ equal or not? Justify your answer.

By the properties of left cosets, two cosets $a + H$ and $b + H$ are equal if and only if $-a + b \in H$. Since $(17/20) - (3/5) = (1/4) \in H$, the cosets $(3/5) + H$ and $(17/20) + H$ are equal.

Alternatively, one could note that $(17/20) = (3/5) + (1/4) \in (3/5) + H$ and $(17/20) = (17/20) + 0 \in (17/20) + H$. Since the cosets $(3/5) + H$ and $(17/20) + H$ are not disjoint, by the properties of cosets they must be equal.