Math 103A Fall 2007 Exam 1

October 31, 2007

NAME:

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Total /100	

Problem 1 (30 points)

1 Let $D_3 = \{R_0, R_{120}, R_{240}, S_1, S_2, S_3\}$ be the dihedral group of order 6, which consists of symmetries of an equilateral triangle. Here, each R_i is the symmetry of the square given by *counterclockwise* rotation by i degrees. Each S_i is a reflection about an axis of symmetry of the triangle, labeled as follows.

(1a) (10 pts). Calculate the product $S_1 S_2$ in the group D_3. Show your work.

(1b) (10 pts). Complete the following Cayley table of the group D_3. (Remember that the product $g_1 g_2$ is written in row g_1 and column g_2 of the Cayley table.) You may freely rely on facts you know about Cayley tables.

	$\mathbf{R_0}$	$\mathbf{R_{120}}$	$\mathbf{R_{240}}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$
$\mathbf{R_0}$	R_0	R_{120}	R_{240}	S_1	S_2	S_3
$\mathbf{R_{120}}$	R_{120}	R_{240}	R_0	S_3	S_1	S_2
$\mathbf{R_{240}}$	R_{240}	R_0	R_{120}	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3			
$\mathbf{S_2}$	S_2	S_3	S_1			
$\mathbf{S_3}$	S_3	S_1	S_2			

(1c) (10 pts). Calculate $Z(D_3)$, in other words, find the *center* of the group D_3. Justify your answer.

Problem 2 (25 points)

(a) (15 pts) Let G be an *Abelian* group with identity element e, and define $H = \{x \in G \,|\, x^2 = e\}$. Prove that H is a subgroup of G.

(b) (10 pts) The point of this part is to show that the result of part (a) need not hold for a non-abelian group. Let $G = D_3$ be the group of symmetries of a triangle (which already appeared in problem 1), and again we define $H = \{x \in G | x^2 = e\}$. Prove that H is *not* a subgroup of G.

Problem 3 (25 points)

(a) (10 pts). Show that the group $U(14)$ is a cyclic group.

(b) (5 pts). Find a subgroup H of $U(14)$ with $|H| = 3$.

(d) (10 pts). Working in the group $U(14)$ still, calculate $[9]^{-100}$. Show your work.

Problem 4 (20 points)

(a) (10 pts). Let $S = \mathbb{R}$ be the set of real numbers, and suppose we define a binary operation on S by the formula $a \star b = a - b$. Is S with this binary operation a group? Either prove it is a group or prove it is not a group.

(b) (10 pts). Let $S = \{x \in \mathbb{R} | x \neq 0\}$ be the set of nonzero real numbers, and define a a binary operation on S by the formula $a \star b = 2ab$. Is S with this binary operation a group? Either prove it is a group or prove it is not a group.