Math 103a Fall 2012 Homework 9

Due Friday December 7 in homework boxes in basement of AP&M

Reading assignment: Chapters 10 and 11.

Exercises related to Chapter 10 and 11:

1. Let G be the dihedral group D_n for some $n \geq 3$. Define a function $f : D_n \to \{1, -1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rotation} \\ -1 & \text{if } x \text{ is a reflection.} \end{cases}$$

Here, consider $\{1, -1\}$ as a group under multiplication. Prove that f is a homomorphism of groups. Find the kernel and image of f, and write down what the first isomorphism theorem says when applied to this homomorphism.

2. Suppose that $\phi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a homomorphism with $\phi([7]_{50}) = [6]_{15}$.
(a). Show that $\phi([7n]_{50}) = [6n]_{15}$ holds for any $n \geq 0$.
(b). Determine $\ker \phi$ and $\operatorname{Im} \phi$, and write down what the first isomorphism theorem says when applied to this particular homomorphism.
(c). Find the set of all $[b]_{50} \in \mathbb{Z}_{50}$ such that $\phi([b]_{50}) = [3]_{15}$.

3. Is there a homomorphism $\phi : \mathbb{Z}_3 \to \mathbb{Z}_6$ such that $\phi([1]_3) = [5]_6$? Either find such a homomorphism or prove that none exists.

4. Suppose that $\phi : G \to H$ is a homomorphism and that ϕ is a surjective (onto) function. Suppose that H has an element of order d. Prove that G also has an element of order d.

5. Find a homomorphism $\phi : U(30) \to U(30)$ with kernel $\ker \phi = \{[1], [11]\}$ and such that $\phi([7]) = [7]$.

6. Write down a complete list of Abelian groups of order $324 = (2^2)(3^4)$ up to isomorphism.

7. Suppose that G is an Abelian group of order 16, and in computing orders of its elements, you come across an element of order 8 and two elements of order 2. Find which direct product of cyclic groups of prime power order G is isomorphic to. (This problem was changed from its original incorrect version on Tuesday December 4).

8. Recall that an $n \times n$ matrix $A = (a_{ij})$ is called a diagonal matrix if the only nonzero entries of the matrix are the entries along the main diagonal; that is, $a_{ij} = 0$ unless $i = j$. Let G be the group of all $n \times n$ diagonal matrices whose diagonal entries lie in the set $\{-1, 1\}$, with operation multiplication of matrices. Note that since there are two possibilities for each of n diagonal entries, $|G| = 2^n$.

Show that G is an Abelian group and then find, with proof, which direct product of cyclic groups of prime power order is isomorphic to G.