Math 103a Fall 2012 Homework 8

Due Friday November 30 in homework boxes in basement of AP&M

Reading assignment: Finish reading chapter 9, and read Chapters 10 and 11.

Exercises related to Chapter 9:

1. Let $G = \mathrm{GL}(2, \mathbb{R})$ and let K be a subgroup of $\mathbb{R}^*$, the group of nonzero real numbers under multiplication. Prove that

$$H = \{A \in G \mid \det A \in K\}$$

is a normal subgroup of G.

2. Let $H = \{\epsilon, (12)(34)\}$ in A_4. Show that H is a subgroup of A_4, but that H is not a normal subgroup of A_4. Let $\alpha_1 = (243)$, $\alpha_2 = (142)$, $\alpha_3 = (132)$, and $\alpha_4 = (234)$. Show that $\alpha_1 H = \alpha_2 H$ and $\alpha_3 H = \alpha_4 H$, but that $\alpha_1 \alpha_3 H \neq \alpha_2 \alpha_4 H$. Explain why this means that the left cosets of H do not form a group under coset multiplication (i.e., there is not a well-defined factor group G/H.)

3. What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$?

4. Let $Z = \{R_0, R_{180}\} \subseteq D_6$. Z is known to be the center of the group D_6, which means Z must be a normal subgroup of D_6 (assume this). What is the order of the element $R_{60}Z$ in the factor group D_6/Z?

5. Let $G = U(32)$ and $H = \{[1], [31]\} = \langle [31] \rangle \subseteq G$. The group G/H is isomorphic to one of the following groups: $\mathbb{Z}_8, \mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one it is.

6. Let G be cyclic group, and let H be any subgroup of G. Prove that H is normal in G, and that the factor group G/H is also a cyclic group.

7. Let a be an element of a finite group G, with order $|a| = m$. Let H be a normal subgroup of G. Consider the element aH in the factor group G/H. Prove that the order of aH in G/H is a divisor of m.

8. Let G be a group with $|G| = pq$, where p and q are primes that are not necessarily distinct. Let $Z(G)$ be the center of G. Prove that either $Z(G) = \{e\}$ is trivial or else $Z(G) = G$. (Hint: G/Z-theorem).

9. Let G be the group $\mathbb{R}^*$ of all nonzero real numbers under multiplication. Suppose that H is a subgroup of G with $|G : H| = 2$, that is, there are precisely two distinct left cosets of H in G. Prove that H must be equal to $\{a \in \mathbb{R}^* | a > 0\}$, the subgroup of $\mathbb{R}^*$ consisting of all positive real numbers. (Hint: consider the factor group G/H. If a is a positive real number, then $a = b^2$ for some other real number b (a square root). Which of the two cosets of G/H must b^2H be?)