Math 103a Fall 2012 Homework 5

Due Friday 11/2/2012 by 4pm in homework box in Basement of AP&M

Reading assignment: Continue to read Chapters 6-7 of Gallian, and read Chapter 5, which will be the topic of the next homework after this one.

Exercises related to Chapter 6

In the first few exercises below, an *automorphism* is an isomorphism $\phi : G \to G$ from a group G to the same group G. Although one uses this different name to emphasize that the domain and target of the function are the same group, one proves a function is an automorphism in the same way one proves a function is an isomorphism; there is no difference other than the name.

1. Consider the set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of all nonzero complex numbers, which is a group under multiplication of complex numbers. Show that the complex conjugation function $\phi : \mathbb{C}^* \to \mathbb{C}^*$ with formula $\phi(a + bi) = a - bi$ is an automorphism of $\mathbb{C}^*$.

(Review complex numbers first — in particular how to multiply them— if you feel rusty on this subject.)

2. Let G be a group. Fix an element $x \in G$. Prove that the function $\phi : G \to G$ given by $\phi(g) = xgx^{-1}$ for all $g \in G$ is an automorphism of G. (ϕ is called "conjugation by x").

3. Let G be a group. Prove that the function $\phi : G \to G$ given by $\phi(g) = g^{-1}$ for all $g \in G$ is an automorphism if and only if G is Abelian.

4. Prove that the group $\mathbb{Z}$ of integers under addition is not isomorphic to the group $\mathbb{Q}$ of rational numbers under addition.

Exercises related to Chapter 7

5. Find all distinct left cosets of the subgroup $H = \{[1], [4]\}$ of $U(15)$. How many are there?

6. Suppose that a has order 15. Find all of the distinct left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$. How many are there?

7. Suppose that H and K are subgroups of a group G. if $|H| = 12$ and $|K| = 35$, find $|H \cap K|$.

8. Let a and b be nonidentity elements of different orders in a group G of order 155. Prove that the only subgroup of G that contains both a and b is G itself.

9. Given a positive integer n, the number $\phi(n)$ is defined to be the number of integers a with $0 \leq a \leq n-1$ such that $\gcd(a, n) = 1$. By the definition of the group $U(n)$, $\phi(n)$ is the number of elements in the group $U(n)$. The function ϕ is called the *Euler phi function* and is discussed at the end of Chapter 4 in the text, but we did not discuss it in class. You don't need to know anything special about it to do this problem.

Prove that if a and n are positive integers with $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$. (Hint: mimic the proof of Fermat's little theorem.)

10. Let $|G| = 8$. Show that G must have an element of order 2.

11. Suppose that G is a group with more than one element and that G has no subgroups except $\{e\}$ and G. Prove that $|G|$ is finite, and then prove that $|G|$ is a prime number.