# Math 103a Fall 2012 Homework 2

Due Friday 10/12/2012 by 4pm in homework box in Basement of AP&M

**Reading assignment:** Chapters 1- 3 of Gallian. Notice that in class we are covering Chapter 2, then Chapter 1, then Chapter 3.

**Exercises related to Chapter 2**

1. Let $M_2(\mathbb{R})$ be the set of all $2 \times 2$ matrices with entries in $\mathbb{R}$. Define

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \middle| \det(A) = ad - bc = 1 \right\}.$$

Show that $\mathrm{SL}_2(\mathbb{R})$ is a group under the operation of matrix multiplication. (It is called the *special linear group*).

2. Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}$. Show that $G$ is a group under the operation of matrix multiplication. In linear algebra, you learn that a matrix is invertible if and only if its determinant is not 0. Explain why the fact that the matrices all have determinant 0 in the group $G$ does not contradict axiom (3) that every element of a group has an inverse.

For the next two problems, recall that $U(n) = \{[a]_n | \gcd(a, n) = 1\} \subseteq \mathbb{Z}_n$ is a group under matrix multiplication, and is called the units group of $\mathbb{Z}_n$.

3. Write down a complete multiplication table (Cayley Table) for $U(9)$.

4. For any integer $n > 2$, show that there are at least two elements $x \in U(n)$ that satisfy $x^2 = 1$. If $n > 2$ is prime, show that there are *exactly* two such elements $x$.

5. Let $G$ be a group with the following property: whenever $a, b, c \in G$ and $ab = ca$, then $b = c$. Prove that $G$ is Abelian. (Thus, in a non-Abelian group, you should not cancel an element from two different sides.)

6. Give an example of a group with 105 elements. Give two different examples of groups with 44 elements.

7. Prove that the set $\{3^m 6^n | m, n \in \mathbb{Z}\}$ is a group under the operation of multiplication of rational numbers.

8. Prove that if $G$ is a group with the property that $x^2 = e$ for all $x \in G$, then $G$ is Abelian.

**Exercises related to Chapter 1**

9. Describe in pictures or words the 10 elements of $D_5$, the symmetries of a regular pentagon. I suggest the following notation, similar to the notation for $D_4$ in the text. First, label the vertices of the pentagon with the numbers 1-5. For a rotation, call it $R_\theta$ where $\theta$ is the angle of the counterclockwise rotation. For a reflection, call it $S_i$ where the axis of reflection goes through the vertex you label with the integer $i$.

Now write down a full multiplication table of $D_5$, similar to the multiplication table for $D_4$ given in Chapter 1 of the text. This has many entries (100), but there are many patterns to the entries that once you realize the pattern, should make the filling in of these entries quite fast. But to start, you will have to calculate some products by hand by drawing pictures.

(You do not need to justify your answer for this one. The point of this exercise is to get a lot of practice with multiplying elements in a dihedral group so that you understand how this group works.)

10. Let $D_n$ be the dihedral group of symmetries of a regular $n$-gon in the plane, where $n$ is now any integer with $n \geq 3$. You may assume without proof that this group has $n$ reflections and $n$ rotations.

Explain geometrically why the following facts are true:

(i) Any reflection followed by another reflection is a rotation.

(ii) Any rotation followed by another rotation is a rotation.

(iii) Any rotation followed by a reflection, or a reflection followed by a rotation, is a reflection.