Math 103a Fall 2012 Homework 1

Due Friday 10/5/2012 by 4pm in homework box in Basement of AP&M

Reading assignment: Read Chapters 0-2 of Gallian. This first set of exercises is related entirely to Chapter 0 and consists of problems reviewing Math 109 material.

Exercises:

1. Let a, b be integers and $d = \gcd(a, b)$. If $a = da'$ and $b = db'$ for some integers a', b', show that $\gcd(a', b') = 1$.

2. Let a and b be positive integers and $d = \gcd(a, b)$ and $m = \operatorname{lcm}(a, b)$. If t divides both a and b, prove that t divides d. If s is a multiple of both a and b, prove that s is a multiple of m.

3. Generalize Euclid's Lemma by proving the following: If p is a prime number and p divides a product of integers $a_1 a_2 \ldots a_n$, then p divides a_i for some i. (This is proved for $n = 2$ only in the text and in class).

4. Use the generalized Euclid's Lemma of exercise 3 to prove the uniqueness portion of the fundamental theorem of arithmetic.

5. Suppose that a, b are positive integers and that a and b have prime factorizations $a = p_1^{e_1} \ldots p_m^{e_m}, b = p_1^{f_1} \ldots p_m^{f_m}$. Here, we assume that the p_i are distinct primes, and we allow $e_i \geq 0, f_i \geq 0$ so that we can assume the same sets of primes occur in both factorizations. (For example, if $a = 15$, $b = 12$, then $a = 2^0 3^1 5^1$ and $b = 2^2 3^1 5^0$).

Show that $\gcd(a, b) = p_1^{g_1} \ldots p_m^{g_m}$, where $g_i = \min(e_i, f_i)$ for each i. Show also that $\operatorname{lcm}(a, b) = p_1^{h_1} \ldots p_m^{h_m}$, where $h_i = \max(e_i, f_i)$ for each i. Finally, prove that

$$\gcd(a, b) \operatorname{lcm}(a, b) = ab.$$

(Hint: use the fundamental theorem of arithmetic).

6. Determine $2^{500} \bmod 7$ (in the notation of the book). In other words, (in the notation I prefer), find the unique r with $0 \leq r < 7$ such that $2^{500} \equiv r \pmod 7$.

7. Prove that $n^3 \equiv n (\bmod 6)$ holds for all integers n (not just positive integers.)

8. Recall that the Fibonacci sequence is $1, 1, 2, 3, 5, 8, \ldots$ where we set $f_1 = 1, f_2 = 1$, and for $n \geq 3$, we inductively define $f_n = f_{n-1} + f_{n-2}$. Prove for all $n \geq 2$ that f_n is even if and only if n is a multiple of 3.

9. Let S be the set of all real numbers. If $a, b \in S$, define $a \sim b$ if $a - b$ is an integer. Prove that $\sim$ is an equivalence relation on S. Describe what the partition of S into equivalence classes looks like.

10. Let $S = \mathbb{R}^2$ be the set of all points (x, y) in the real cartesian plane. Define a relation by $(x, y) \sim (w, z)$ if $x + y = w + z$. Prove that this is an equivalence relation on S. Describe geometrically what the partition of S into equivalence classes looks like.