Math 103A Fall 2012 Exam 2 Solutions

November 14, 2012

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Problem 1 (10 points)

Consider the element $\alpha = (541)(3742)(1265) \in S_7$.

(a). (3 pts) Write α in disjoint cycle form.

$\alpha = (137)(264)$. (There are other variations that are correct, for example $\alpha = (642)(713)$ or $\alpha = (642)(713)(5)$.)

(b). (2 pts) Find the order of α. Explain your answer in a sentence.

By a theorem in Chapter 5, the order of a permutation written in disjoint cycle form is the least common multiple of the lengths of the cycles. Thus $|\alpha| = \text{lcm}(3,3) = 3$ (or $\text{lcm}(3,3,1) = 3$ if you wrote the 1-cycle.)

(c). (3 pts) Is $\alpha \in A_7$? Justify your answer.

We have $\alpha = (137)(264) = (13)(37)(26)(64)$. Since α is a product of an even number of 2-cycles, α is an even permutation and since A_n consists of the even permutations by definition, the answer is yes, $\alpha \in A_7$.

(d). (2 pts) Find α^{-1}, expressing it also in disjoint cycle form.

The inverse of a cycle is found by reversing the numbers in the cycle, for example $(137)^{-1} = (731)$. Thus $\alpha^{-1} = (264)^{-1}(137)^{-1} = (462)(731)$. (Again, there are lots of other ways to write this that are also correct.)

Problem 2 (10 points)

(a) (7 pts) Find an isomorphism $\phi : \mathbb{Z}_6 \to U(7)$. Explain why you know that ϕ is an isomorphism.

We have $\mathbb{Z}_6 = \{[0], [1], [2], [3], [4], [5]\}$ (congruence classes modulo 6) and $U(7) = \{[1], [2], [3], [4], [5], [6]\}$ (congruence classes modulo 7).

By results in chapter 6, a finite group G of order n is isomorphic to Z_n if and only if it is cyclic, and in this case, an isomorphism is found by sending $[1] \in Z_n$ to a generator a of G.

Thus to find an isomorphism we need to know what is a generator of $U(7)$. By trying different one elements one finds that $|[3]| = 6$ and so $\langle [3] \rangle = U(7)$. So $[3]$ is a generator of $U(7)$ and the function $\phi : \mathbb{Z}_6 \to U(7)$ given by $\phi([i]) = [3]^i$ for each $[i] \in \mathbb{Z}_6$ is an isomorphism by the results in chapter 6.

(b) (3 pts) Is ϕ is the only such isomorphism, or does there also exist a different isomorphism $\psi : \mathbb{Z}_6 \to U(7)$? Justify your answer.

If $U(7)$ has any other generators, we can write a different isomorphism by using a different generator. By trying each element, one finds that $[5]$ is the only other generator of $U(7)$. Or, one can use the theorem that if $\langle a \rangle$ is cyclic of order n, then the generators are the elements of the form a^j where $\gcd(j, n) = 1$. Thus taking $a = [3]$ and $n = 6$ we find that $a^1 = [3]$ and $a^5 = [3]^5 = [5]$ are the only generators.)

Thus there is another isomorphism by the results in chapter 6, namely $\psi : \mathbb{Z}_6 \to U(7)$ given by $\phi([i]) = [5]^i$.

(Remark: this problem is essentially the same problem as the final homework problem on homework #4: the only difference is the homework problem was about $U(9)$ instead of $U(7)$. That homework problem was even graded. So if you had no idea how to do this problem on the exam, take this as a lesson that you should go back over your homework and figure out how to do those problems that you did not do correctly the first time.)

Problem 3 (10 points)

This exercise is about the groups S_4 (the symmetric group of permutations of four numbers) and D_{12} (the dihedral group of symmetries of a regular 12-sided polygon).

(a). (2 pts) Show that the two groups S_4 and D_{12} have the same number of elements. (You shouldn't have to do any lengthy calculations for this.)

The formula for the number of elements in S_n is $n!$. So $|S_4| = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. On the other hand, the group D_n always has n rotations and n reflections, so D_{12} has $12 + 12 = 24$ elements also.

(b). (4 pts) Prove that for every element $\alpha \in S_4$, the order of α is less than or equal to 4.

This is similar to the problem on the homework where you needed to find the largest possible order of an element of A_7, but easier. The order depends on the lcm of the lengths of the cycles in disjoint cycle form, so you have to look at every possible disjoint cycle form among elements in S_4. The possible disjoint cycle forms are $\epsilon, (ab), (ab)(cd), (abc)$, or $(abcd)$, where a, b, c, d indicate unknown but different numbers between 1 and 4. The orders of elements like this are $1, 2, 2, 3$, or 4, respectively. Thus 4 is the maximum possible order for elements of S_4.

(c). (4 pts) Show that S_4 and D_{12} are not isomorphic.

The group D_{12} has an element R_{30} which rotates 1/12 of the way around the circle, that is 30 degrees. Since this operation needs to be performed 12 times until one obtains the identity element of D_{12}, the order of R_{30} is 12. Since S_4 has no elements with order 12 by part (b), and isomorphic groups have the same number of elements of each order, S_4 and D_{12} cannot be isomorphic.

Problem 4 (10 points)

Suppose that G is a group with $|G| = 49$ in this problem.

(a). (3 pts) Suppose that $a \in G$. What are the possible values for $|a|$? Justify your answer.

By Lagrange's theorem, the only possible orders for elements of a group of order 49 are the possible divisors of 49, that is $1, 7$, or 49.

(b). (3 pts) Suppose that G is not cyclic. Count the number of elements of G which have order 7. Justify your answer.

There is no obvious way to count the elements of order 7; the trick is to count the number of elements of order 1 and order 49, and thus determine the number of elements of order 7 by subtraction. The only element of order 1 (in any group) is the identity element. If G had an element a of 49, then $\langle a \rangle$ would be a subgroup with 49 elements, and thus equal to all of G, so G would be cyclic. By assumption G is not cyclic, thus G has no elements of order 49. Using part (a), this forces all 48 non-identity elements of G to have order 7, so the answer is 48.

(c). (4 pts) Suppose that G *is* cyclic. Count the number of elements of G which have order 7 in this case. Justify your answer. (Hint: recall that if b has order n in a group, then b^i has order $\dfrac{n}{\gcd(i, n)}$ in the group.)

Since G is cyclic, $G = \langle b \rangle$ for some element b which must have order 49. Then an arbitrary element of G is of the form b^i, and thus has order $49/\gcd(i, 49)$ by the formula given in the hint. So we want to know which i with $0 \leq i < 49$ will give us $49/\gcd(i, 49) = 7$. So we want $\gcd(i, 49) = 7$. It is easy to see the solutions are $i = 7, 14, 21, 28, 35, 42$. So there are six elements of order 7, namely $\{b^7, b^{14}, b^{21}, b^{28}, b^{35}, b^{42}\}$.

(Actually, there is a formula in Chapter 4 for the number of elements of order d in a cyclic group of order n, where d divides n: the formula is $\varphi(d)$, where φ is Euler phi function. So the answer is $\varphi(7) = 6$. Since we did not cover this formula in class, I didn't expect you to do it this way.)