MATH 100C FALL 2016 MIDTERM REVIEW SHEET

Your exam is Monday May 2 in 6402 AP&M. The room is booked from 12pm-3pm. You get one hour and 50 minutes, so come either from 12-1:50pm or from 1-2:50pm. The exam covers Chapter 6, sections 6.1-6.6, and homeworks 1-4. Please bring a blue book.

Some of the problems on the exam will be similar in style to homework problems. I will also ask you to reproduce the proofs of a sample of key theorems we proved. These will be chosen from among the results I list below. (Note that you are not going to be asked to prove all of these on the test, just a few of them.) So you should study the results below, and also study homeworks 1-4.

1. Prove that if $K \subseteq F$ is a field extension and $u \in F$ is an algebraic element, then $K(u) \cong K[x]/\langle f(x) \rangle$ as rings, where $f(x)$ is the minimal polynomial of u over K.

2. Prove Kronecker's theorem: if K is a field and $f(x) \in K[x]$, then there is a field extension $K \subseteq F$ such that $f(u) = 0$ for some $u \in F$.

3. Prove that if $K \subseteq F$ is a field extension, then the set $E = \{a \in F | a \text{ is algebraic over } K\}$ is a subfield of F.

4. Prove that if $K \subseteq E \subseteq F$ and $K \subseteq E$ and $E \subseteq F$ are algebraic extensions, then $K \subseteq F$ is an algebraic extension.

Date: April 27, 2016.

5. Recall the theorem that a real number u is constructible if and only if there is a sequence of real numbers $u_1, \ldots, u_n$, such that $u \in \mathbb{Q}(u_1, \ldots, u_n)$ with $u_i^2 \in \mathbb{Q}(u_1, \ldots, u_{i-1})$ for all $1 \leq i \leq n$ (when $i = 1$ this is interpreted as $u_1^2 \in \mathbb{Q}$.)

Prove using this theorem that if u is a constructible number, then $[\mathbb{Q}(u) : \mathbb{Q}]$ is a power of 2.

6. Prove that if K is a field and $f(x) \in K[x]$, then there exists a splitting field F for $f(x)$ over K.

7. Prove that if F is a field of characteristic p and $n \geq 1$, then $E = \{a \in F | a^{p^n} = a\}$ is a subfield of F.

8. Prove that for any $n \geq 1$ and prime number p, the splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$ is a field F with p^n elements, and that in fact F is precisely the set of roots of $x^{p^n} - x$ in F.

9. Prove that if F is a field with p^n elements, where p is prime, then F is the splitting field over its prime subfield $\mathbb{Z}_p$ of the polynomial $x^{p^n} - x$. Conclude that there is only one field with p^n elements up to isomorphism.