Math 100c Spring 2016 Homework 7

Due Friday 5/27/2016 by 3pm in HW box in basement of AP&M

Reading

Read Section 8.4 in the text. We will cover some concepts on solvable groups from 7.6 and 7.7 also, but you are not responsible for the bulk of those sections.

Problems from the text (write up full solutions):

Section 8.5 #3, 4, 9, 10(b)(d)

(remarks: do #10(d) before #4. For #9, use the multiplicative form of the Mobius inversion formula from Theorem 6.6.8).

Problems not from the text (write up full solutions):

A. Use the result of #9 above to calculate $\Phi_{36}(x)$.

B. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ be complex conjugation $z = a + bi \mapsto \overline{z} = a - bi$. Let $n \geq 3$, let $\zeta = e^{2\pi i/n}$ be a primitive nth root of unity, and let $F = \mathbb{Q}(\zeta) \subseteq \mathbb{C}$ be the splitting field of $x^n - 1$ over $\mathbb{C}$.

(a). Show that for any $i \in \mathbb{Z}$, $\overline{\zeta^i} = \zeta^{-i} = \zeta^{n-i}$. Conclude that σ restricts to an automorphism θ of F.

(b). Let $H = \langle \theta \rangle$ be the subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ generated by θ. Show that since $n \geq 3$, $|H| = 2$. Let $E = F^H$ be the fixed field of H and show that $\zeta + \zeta^{-1} \in E$.

(c). Show that $E = \mathbb{Q}(\zeta + \zeta^{-1})$. (Hint: Show that ζ satisfies a degree 2 polynomial with coefficients in $\mathbb{Q}(\zeta + \zeta^{-1})$.)

(d). Show that $E = F \cap \mathbb{R}$; that is, E consists of those elements of F that are real numbers.

C. Prove that there is no root of unity ζ such that $\mathbb{Q}(\sqrt[3]{2})$ is contained in $\mathbb{Q}(\zeta)$.