Math 100c Spring 2016 Homework 6

Due Friday 5/20/2016 by 3pm in HW box in basement of AP&M

Reading

Read Section 8.3 and begin to read Section 8.5 in the text (we will skip 8.4 and come back to it)

Problems not from the text (write up full solutions):

In problems A-B below, for each of the following polynomials $f(x) \in Q[x]$, consider the splitting field F of $f(x)$ over $\mathbb{Q}$. In each case, identify the Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$ as a familiar group. For each problem, draw a diagram similarly as in Figure 8.3.1 or Figure 8.3.2 in the text which exhibits all of the subgroups H of G and all fields E with $\mathbb{Q} \subseteq E \subseteq F$. Make it clear which fields correspond to which subgroups. Also, draw on each line of the diagram a number indicating the degree of the extension (in the field diagram) or the index of the subgroup (in the subgroup diagram). Justify your answers.

A. $f(x) = x^4 - x^2 - 6$.

B. $f(x) = x^7 - 1$ (Hint: to explicitly describe the intermediate subfields, try writing them as $\mathbb{Q}(u)$ where u is a carefully chosen sum of powers of a primitive 7th root of unity ζ).

Problems C-E are all related to the following example: Let $f(x) = x^6 - 2$ and let F be the splitting field of $f(x)$ over $\mathbb{Q}$.

C. Show that $G = \mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to the dihedral group of order 12.

(Hint: find a and b in the Galois group with $a^6 = e$, $b^2 = e$ and $ba = a^{-1}b$, which is enough to show the group is dihedral, as in example 3.6.3 in the text.)

D. Find the proper nontrivial subgroups of G which are normal in G and prove they are the only ones (there is one of order 2, one of order 3, and three of order 6).

Note: this was adjusted on May 17; the original version incorrectly stated that there is only one normal subgroup of order 6.

E. For each of the three proper nontrivial normal subgroups H of G in problem D, find the corresponding fixed field F^H explicitly. By the fundamental theorem, $\mathbb{Q} \subseteq F^H$ is a normal extension; find an explicit polynomial $g(x)$ such that F^H is the splitting field of $g(x)$ over $\mathbb{Q}$. Justify your answers.

F. Let V be a finite-dimensional vector space over an infinite field K. Suppose that $U_1, \dots, U_n$ are proper K-subspaces of V, so $U_i \subsetneq V$ for all i. Show that $V \neq U_1 \cup U_2 \cup \cdots \cup U_n$.

(suggested outline: Suppose the result is false. One may assume that n is the smallest integer such that V can be written as a union of n proper subspaces. Clearly $n > 1$. Thus we have $V = U_1 \cup U_2 \cup \cdots \cup U_n$, but $V \neq U_2 \cup \cdots \cup U_n$. Pick $u \in V$ such that $u \notin U_2 \cup \cdots \cup U_n$, and pick any $v \in V$ with $v \notin U_1$. Consider the elements $w = u + \lambda v$ as $\lambda \in K$ varies; each of these must be contained in one of the U_i, but show that no U_i can contain more than one of them, and use that there are infinitely many λ to choose from to get a contradiction).

G. Let $K \subseteq F$ be a field extension with $[F : K] < \infty$. Suppose that there are only finitely many subfields E such that $K \subseteq E \subseteq F$. Show that $F = K(u)$ for some $u \in F$. (Hint: when K is a finite field, use the structure theory of finite fields. When K is infinite, use problem F.)

H. Let $F = \mathbb{Q}(\sqrt[3]{2}, \zeta)$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$, where $\zeta = e^{2\pi i/3}$. Find an explicit element $u \in F$ such that $F = \mathbb{Q}(u)$.

(Hint: The fundamental theorem of Galois theory tells you what the intermediate fields E with $\mathbb{Q} \subseteq E \subseteq F$ are.)

Optional problem (don't hand in)

Find the complete subgroup and corresponding subfield diagrams for the splitting field of $x^6 - 2$ over $\mathbb{Q}$, as in problems A-B (problems C-E only asked you to look at the normal subgroups of this example)