Math 100c Spring 2016 Homework 5

Due Friday 5/13/2016 by 3pm in HW box in basement of AP&M

Reading

Read Sections 8.1-8.2 and begin to read Section 8.3 in the text.

Assigned Problems from the text (write up full solutions):

Section 8.1: #5, 6, 7

Section 8.2 #1, 4, 5

Addtional problems not from the text (write up full solutions):

A. Suppose that $f(x) \in K[x]$ is a monic irreducible polynomial of degree n. Let F be a splitting field for $f(x)$ over K, and suppose that $f(x) = (x - r_1)(x - r_2) \dots (x - r_n) \in F[x]$, where the roots $r_1, \dots, r_n$ of $f(x)$ in F are distinct. As we have seen, if $\theta \in \mathrm{Gal}(F/K)$ then θ permutes the set $\mathcal{R} = \{r_1, \dots, r_n\}$ of roots of $f(x)$ in F. In other words, any such θ gives a permutation $s_\theta \in S_n$, where S_n is the symmetric group on $\{1, \dots, n\}$, by tracking the subscripts of how θ moves $\mathcal{R}$ around; explicitly, $s_\theta(i) = j$ if $\theta(r_i) = r_j$.

Show that the function $f : \mathrm{Gal}(F/K) \to S_n$ given by $\theta \mapsto s_\theta$ is a homomorphism of groups which is always injective (one-to-one). Conclude that the group $\mathrm{Gal}(F/K)$ is isomorphic to a subgroup of S_n.

B. Show that the Galois group of $x^3 - 5$ over $\mathbb{Q}$ is isomorphic to S_3. (Hint: use problem A).

C. Consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[4]{2})$, where $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ since the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$ is $x^4 - 2$.

Consider the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q})$. How many elements does this group have?

D. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be an automorphism of the real numbers. This problem assumes a little bit of analysis (From Math 140a or Math 142a).

(a). Prove that $\sigma(q) = q$ for any rational number $q \in \mathbb{Q}$.

(b). Prove that σ takes squares to squares and hence takes the set of positive real numbers to itself. Conclude that $a < b$ implies that $\sigma(a) < \sigma(b)$.

(c). Prove that $-1/m < a - b < 1/m$ implies that $-1/m < \sigma(a) - \sigma(b) < 1/m$ for every positive integer m. Conclude that σ is a continuous function.

(d). Prove that since σ is continuous and fixes the set of rational numbers which is dense in $\mathbb{R}$, that σ is just the identity map on $\mathbb{R}$. Hence $\mathrm{Aut}(\mathbb{R})$ is the trivial group.