Math 100c Spring 2016 Homework 2

Due Friday 4/15/2016 by 3pm in HW box in basement of AP&M

Reading

All references are to Beachy and Blair, 3rd edition.

Read Sections 6.3-6.4.

Assigned Problems from the text (write up full solutions and hand in):

Section 6.1: #7, 9

Hint for #9: use #7 and the results of Section 6.2.

Section 6.2: #7

Section 6.3: #3 (Hint: Recall that $(x^n - 1) = (x-1)(x^{n-1} + x^{n-2} + \cdots + 1)$. Thus if $\zeta^n = 1$ and $\zeta \neq 1$, then $1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0$.

Addtional problems not from the text (write up full solutions and hand in):

A. (a). Let K be a field of characteristic not equal to 2. Suppose that $K \subseteq F$ is a field extension such that $[F : K] = 2$. Prove that there is some $\beta \in F$ such that $\beta^2 \in K$ and such that $F = K(\beta)$. In other words, any extension of degree 2 of K can be generated by an element which is a square root of an element of K. (Hint: first show that $F = K(\alpha)$

for some element α, where $f = \text{minpoly}_K(\alpha)$ has degree 2. Then think about how α can be expressed in terms of the coefficients of f.)

(b). Let $K = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with 2 elements and consider $F = K[x]/(x^2+x+1)$. Show that F is a field with exactly four elements. As usual we can think of K as a subfield of F, so that we have a field extension $K \subseteq F$. Show that $[F : K] = 2$, but there is no element $\beta \in F$ such that $F = K(\beta)$ and $\beta^2 \in K$.

B. Let $F \subseteq \mathbb{R}$ be the field of constructible numbers, which we defined to be those $a \in \mathbb{R}$ such that the length $|a|$ occurs as the distance between two points in $\mathbb{R}^2$ constructible using compass and straightedge.

(a). Show that the points in $\mathbb{R}^2$ which are constructible using compass and straightedge are exactly the points (a, b) where $a, b \in F$.

(b). Identify the complex numbers $\mathbb{C}$ with the real plane $\mathbb{R}^2$, where $a + bi$ is identified with (a, b) as usual. Define a complex number $u = a + bi$ to be constructible if (a, b) is a constructible point. Show that if u is a constructible complex number, then $[\mathbb{Q}(u) : \mathbb{Q}]$ is a power of 2.

(c). If a regular n-gon is constructible, show that the primitive nth root of unity $\zeta = e^{2\pi i/n}$ is a constructible number and therefore that $[\mathbb{Q}(\zeta) : \mathbb{Q}]$ is a power of 2.

(d). Show that a complex number in polar form $u = re^{i\theta}$, where $0 \leq r \in \mathbb{R}$ and $\theta \in \mathbb{R}$, is constructible if and only if r is a constructible real number and θ is a constructible angle.

C. (This is an expanded version of 6.3 #4).

Let $\zeta = \cos(2\pi/7) + i\sin(2\pi/7) = e^{2\pi i/7}$ be a primitive 7th root of unity in $\mathbb{C}$.

(a). Find a degree 3 polynomial $f(x) \in \mathbb{Q}[x]$ that $\omega = (\zeta + \zeta^{-1})$ satisfies.

(b). Show the polynomial f you found in (a) is irreducible.

(c). Show that ζ is not constructible as a complex number as defined in problem B.

(d). Show that the regular heptagon (7-sided polygon) is not constructible.

(e). Recall that $x^6 + x^5 + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ by using the Eisenstein criterion after a substitution trick (Corollary 4.4.7). Conclude that $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 6$.

(f). Show that $[\mathbb{Q}(\zeta) : \mathbb{Q}(\omega)] = 2$. Find the minimal polynomial of ζ over $\mathbb{Q}(\omega)$.