Math 100b Winter 2010 Homework 8

Due 3/12/09 in class, or by 5pm in HW box on 6th floor of AP&M

Throughout this homework, let F be an arbitrary field.

1. Let $A = (a_{ij}) \in M_n(F)$ be an arbitrary $n \times n$ matrix. Recall that the *transpose* of A is the matrix A^t we get by flipping A around its main diagonal; in other words, $A^t = (a_{ji})$. For example, $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}^t = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$.

Prove that $\det A = \det A^t$.

Hint/Remark: I suggest you work directly from the definition of the determinant to prove this. This result is useful, because it means that all theorems we proved about columns in class have analogs for rows. For example, the determinant is multilinear and alternating as a function of the rows of a matrix.

2. Let $A = (a_{ij}) \in M_n(F)$ be an arbitrary $n \times n$ matrix. Prove that if you add a scalar multiple of one column to some other column of A, the determinant remains unchanged.

3. Let $A, B \in M_n(F)$ be arbitrary $n \times n$ matrices. In this problem, you will supply the details we omitted in class in the proof of the important result that the determinant is multiplicative, i.e. that $\det(AB) = \det(A)\det(B)$.

Suppose we fix the matrix A but let B vary. Consider the two functions $f, g : M_n(F) \to F$ where $f(B) = (\det A)(\det B)$ and $g(B) = \det(AB)$. If we prove that $f = g$ then the problem will be solved (since the fixed matrix A was arbitrary).

We proved the following theorem in class (3/5):

Theorem 0.1 *Let $h : M_n(F) \to F$ be a function such that h is both multilinear and alternating (when considered as a function of the n column vectors making up the matrix). Then there is $c \in F$ such that $h(B) = c\det(B)$ for all matrices B, in other words h is a scalar multiple of the determinant function. Moreover, $c = h(I)$ where I is the identity matrix.*

Now use this theorem to prove that $f = g$.

Hint: We showed in class that the function det is both multilinear and alternating when considered as a function of the columns of the matrix. Using this, show that both f and g are also multilinear and alternating when considered as functions of the columns. For g, first make the following preliminary observation: If we write $B = \left(v_1 \ \vdots \ v_2 \ \vdots \ \ldots \ \vdots \ v_n\right)$ so that the columns of B are the vectors v_i, then $AB = \left(Av_1 \ \vdots \ Av_2 \ \vdots \ \ldots \ \vdots \ Av_n\right)$, in other words the columns of AB are the vectors Av_i. This means that as a function of the columns, $g(v_1, \ldots, v_n) = \det(Av_1, \ldots, Av_n)$.

4. Let $A = \begin{pmatrix} 4 & 1 & -2 \\ 3 & -5 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ in $M_3(\mathbb{Q})$. Calculate A^{-1} using the cofactor formula for an inverse that we developed in class on Monday 3/8. The point of this is just for you to see this formula in action; there is no proof to be done here. Remember to check that the matrix you find really is the inverse of A, since it is easy to make calculation errors.

(Recall: let A_{ij} be the 2×2 matrix obtained by removing row i and column j from A, then define $c_{ij} = (-1)^{i+j} \det A_{ij}$, the *ij-cofactor* of A. Let $C = (c_{ij})$ be the matrix of cofactors. Then $A^{-1} = \frac{1}{(\det A)} C^t$ is the cofactor formula for the inverse of A, where C^t is the transpose of C.)

Remark: Just as the definition of the determinant we gave is a lousy way to actually calculate the determinant if a matrix is large, the cofactor formula is usually a lousy way to calculate inverses when a matrix is large. However, the formula is of theoretical interest and is often useful in proofs. We will use it in the last week of class to prove the Cayley-Hamilton theorem.

5. Let V be a vector space of dimension n over F, and suppose we have two (ordered) bases $\mathcal{B}_1 = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{B}_2 = \{w_1, w_2, \ldots, w_n\}$ for V. Write $w_j = \sum_{i=1}^n c_{ij} v_i$, and $v_j = \sum_{i=1}^n d_{ij} w_i$. Let $C = (c_{ij})$ and $D = (d_{ij})$, which are matrices in $M_n(F)$.

(a) Prove that $D = C^{-1}$. In particular, C is invertible.

(b). Let $\phi : V \to V$ be any linear transformation and let $A = M_{\mathcal{B}_1}(\phi)$ and $B = M_{\mathcal{B}_2}(\phi)$ be the matrices corresponding to ϕ with respect to the two different bases. Prove that $A = CBC^{-1}$.

Remarks: Recall that by definition, A is the matrix (a_{ij}) where $\phi(v_j) = \sum_{i=1}^n a_{ij} v_i$, and B is the matrix (b_{ij}) where $\phi(w_j) = \sum_{i=1}^n b_{ij} w_i$.