Math 100b Winter 2010 Homework 3

Due 1/29/09 in class, or by 5pm in HW box on 6th floor of AP&M

Reading

All references will be to Beachy and Blair, 3rd edition.

4.3 and 4.4. You might consider rereading previously assigned sections as well to review. To get ahead on reading, read 5.4.

Extra Problems

These are generally easier or just extra problems you should look over to check your understanding of the material. They are not to be handed in.

Section 5.3: 1, 3, 6, 7, 10, 11, 13, 16.

Assigned Problems

Write up neat solutions to these problems.

Section 4.3: 22.

Section 5.2: 23.

Section 5.3: 12, 18, 19, 24.

Hints/comments:

5.2 #23. Use the binomial theorem, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$, which works in any commutative ring as long as we interpret each term $\binom{n}{i} a^i b^{n-i}$ to mean the sum of $\binom{n}{i}$ copies of $a^i b^{n-i}$.

5.3 #12 The binomial theorem will be useful again. For intuition about nilpotent elements (integral domains don't have any except 0), consider $\mathbb{Z}_{p^n}$ as an example, where p is a prime and $n \geq 2$; these rings have many nilpotent elements.

5.3 #18: in part (a), recall that J/I is defined to be the set of cosets $\{x + I | x \in J\}$.

5.3 #24: Note that this problem shows that $\mathbb{Z}[x]$ is not a PID. You might want to do additional problem #1 first: the ideal in question is simply the ideal generated by 2 and x.

Additional Problems

1. Let R be a commutative ring. Given any subset $X \subseteq R$ (possibly infinite), the *ideal generated by* X is defined to be

$$\langle X \rangle = \{a_1x_1 + a_2x_2 + \cdots + a_mx_m | a_i \in R, x_i \in X, m \geq 1\}.$$

In words, this is the set of all possible finite sums of products of elements of X with elements of the ring R. If $X = \{x_1, \ldots, x_d\}$ happens to be finite, we also use notation $Rx_1 + Rx_2 + \cdots + Rx_d$ for $\langle X \rangle$.

(a) Fix a subset X of R. Show that $\langle X \rangle$ is an ideal of R.

(b). Show that $\langle X \rangle$ is the smallest ideal of R which contains every element of X. (in other words, every ideal I of R with $X \subseteq I$ satisfies $\langle X \rangle \subseteq I$.)

2. Recall from problem set 1 the ring R which is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ with pointwise operations: given $f, g \in R$, $f + g$ is the function defined by $[f + g](x) = f(x) + g(x)$, and fg is the function defined by $[fg](x) = f(x)g(x)$.

(a). Given any set S of real numbers, show that $I(S) = \{f \in R | f(x) = 0 \text{ for all } x \in S\}$ is an ideal of R. In words, this is the set of all functions with zeroes at all points in S.

(b). Show that if $S = \{a\}$ consists of a single element, then the ideal $I(S)$ of part (a) is a maximal ideal of R (Hint: show that $R/I(S)$ is isomorphic to a familiar ring in that case.) Show on the other hand that if S consists of more than one element, then $I(S)$ is not even a prime ideal of R.

(c*). (optional) Suppose that $J = \langle f_1, \ldots, f_m \rangle$ is the ideal of R generated by some finite set of functions. Show that $J = I(S)$ for some subset $S \subseteq \mathbb{R}$.

(d*). (optional) Give an example of an infinite list of functions $f_1, f_2, \ldots$ such that the ideal $J = \langle f_1, f_2, \ldots, \rangle$ generated by this set of functions is *not* of the form $I(S)$ for any subset $S \subseteq \mathbb{R}$. (Note that you must make sure that $J \neq R$ as part of this, because $R = I(\emptyset)$.) Together with part (c), this proves that not every ideal of R is can be generated by finitely elements of the ring. (Remark: a ring in which every ideal can be generated by finitely many elements is called *Noetherian* after the mathematician Emmy Noether. So this part shows that the ring R is not Noetherian.)