Math 100b Winter 2010 Homework 1

Due 1/15/09 in class

Reading

All references will be to Beachy and Blair, 3rd edition.

Read Sections 5.1, 4.1, and 4.2 for this week. If you want to get ahead, read 5.2 next.

Extra Problems

These are generally easier or just extra problems you should look over to check your understanding of the material. They are not to be handed in.

Section 5.1: 1, 2(b, c, d), 9, 12, 14, 16.

Section 4.1: 1, 2.

Section 4.2: 1, 2, 3, 4, 5.

Assigned Problems

Write up neat solutions to these problems.

Section 5.1: 2(a, e, f), 4, 7, 14.

(hint for 7: notice first that for any $n \geq 1$, $(1 - a^n) = (1 - a)(1 + a + \cdots + a^{n-2} + a^{n-1})$).

(hint for 14: what is the identity element for multiplication?)

Additional Problems

1. Let R be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. (Note that this is the set of all functions, not just continuous ones.) R becomes a ring if we define the following operations: given $f, g \in R$, $f + g$ is the function defined by $[f + g](x) = f(x) + g(x)$, and fg is the function defined by $[fg](x) = f(x)g(x)$. (These are called *pointwise* operations because the sum and product functions are defined by using the sum and product at each individual point of the real line separately.) It is not hard to show this is a commutative ring; think through this, but don't submit a proof of it. In particular, you should think about what the 0 and 1 elements are in this ring.

(a). Describe the set of units of the ring R.

(b). Describe all elements of the ring R which are zero-divisors.

(recall that a zero-divisor—called a divisor of zero in the text—in a commutative ring S is an element $a \in S$ such that there exists a *nonzero* $b \in S$ with $ab = 0$. So another way to define an integral domain is to say it is a commutative ring such that 0 is the only zero-divisor.)

(c). Suppose we change the definition of multiplication to use *composition* of functions instead, i.e. $[fg](x) = f(g(x))$, and keep the same definition of addition above. Is the set R with these operations a ring? If it is a ring, is it a commutative ring? justify your answers.

2. (a). Suppose that R is an integral domain. Prove that one can cancel nonzero elements from both sides of an equation in R, as follows: if $a \neq 0$ and $ab = ac$, then $b = c$. Give an example of a ring which is not an integral domain and in which the cancelation law above fails.

(b). An *idempotent* of a ring R is an element $a \in R$ such that $a^2 = a$. Show that if R is an integral domain, then 1 and 0 are the only idempotents of R. Give an example of a ring which is not an integral domain and which has more than 2 idempotents.

3. Find all of the units of the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$. (Hint: by definition, $\mathbb{Z}[i]$ is a subring of the complex numbers $\mathbb{C}$. Show that if an element in $\mathbb{Z}[i]$ is a unit in $\mathbb{Z}[i]$, its inverse has to be equal to the usual multiplicative inverse in $\mathbb{C}$.)

4. Let $r \in \mathbb{Q}$ be any rational number such that $\sqrt{r} \notin \mathbb{Q}$, and define

$$\mathbb{Q}[\sqrt{r}] = \{a + b\sqrt{r} | a, b \in \mathbb{Q}\}$$

as a subset of $\mathbb{C}$.

(a). Show that if $a_1 + b_1\sqrt{r} = b_1 + b_2\sqrt{r}$ in $\mathbb{Q}[\sqrt{r}]$, then $a_1 = b_1$ and $a_2 = b_2$.

(b). Prove that $\mathbb{Q}[\sqrt{r}]$ is a subfield of $\mathbb{C}$. (Hint: to check a subset of a field is a subfield, it suffices to prove that it is a subring (so use the subring test) for which every nonzero element is a unit. Given $a + b\sqrt{r} \in \mathbb{Q}[\sqrt{r}]$, it will help to consider the product $(a + b\sqrt{r})(a - b\sqrt{r})$. See also Example 4.1.1 of the text.)

(Notes: We excluded the case $\sqrt{r} \in \mathbb{Q}$, for in that case it is not hard to check that $\mathbb{Q}[\sqrt{r}] = \mathbb{Q}$ and we obtain nothing new. In particular, part (a) fails in that case and part (b) is trivially true.

The fields in this problem are important examples, because they are the simplest examples of fields that are not equal to one of the standard number systems $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. Although it is hard to think up examples at first, there are many interesting examples of fields, and most of Math 100c will be concerned with fields.)