Math 100a Fall 2009 Homework 9

Due 12/4/09 in class or by 4pm in the HW box on the 6th floor of AP&M

Reading

All references are to Beachy and Blair, 3rd edition.

Reading: 7.1 (Theorem 7.1.3 only; the definition of HK is in Section 3.3.), 7.2, 7.3, 7.4, 7.5 (we will not cover most proofs in this section, but read the statement of Theorem 7.5.6 and Example 7.5.1.)

Assigned Problems

Write up neat solutions to these problems:

Section 7.2: 17 (Hint: use exercise 3.8 #14.)
Section 7.3: 1, 2, 11
Section 7.5: 11

Problems not from the text (also to be handed in):

1. Let G be a finite group, and suppose that p is the *smallest* prime dividing $|G|$. (For example, if $|G| = 105$, then $p = 3$.) Prove following the outline below that if G has a subgroup H such that $[G : H] = p$, then H must be a normal subgroup of G.

First let T be the set of left cosets of H in G, so that $|T| = p$. Let G act on the left cosets as in Example 7.3.8 of the text (you also studied that action in Exercise 7.3 #2.) This action gives a homomorphism $\phi : G \to S_p$. Note that $|S_p| = p!$. Exercise 7.3#2(a) shows that $\ker \phi \subseteq H$. Thinking about what numbers divide $p!$, prove that the only possibility is $\ker \phi = H$. Then H is normal because it is the kernel of a homomorphism.

2. Let p be a prime, and let G be a group of order p^2. Prove that either $G \cong \mathbb{Z}_{p^2}$ or else $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(Hint: we proved in class using the class equation that groups of order p^2 are Abelian. This exercise does follow then from the classification theorem of finite Abelian groups, Theorem 7.5.6, but because we are not covering the proof of that theorem I don't want you to use that theorem here. Instead, try the following argument: If G does not have an element of order p^2, then show that you can find normal subgroups H, K of G such that $|H| = |K| = p$ and where $HK = G$, $H \cap K = \{e\}$. Then apply Theorem 7.1.3.)

3. We have now developed enough techniques to classify all groups of order at most 15. Make a list of all groups of order n (up to isomorphism) for each order n with $1 \leq n \leq 15$ *except* $n = 12$, briefly justifying your list. Most of this follows immediately from classification results we have already proved. For example, if $|G| = 2$, then $G \cong \mathbb{Z}_2$ because we proved that any group of prime order is cyclic.

Order 8 is the hardest: first use the fundamental theorem of Abelian groups (Theorem 7.5.6) to classify the Abelian ones (of which there are three non-isomorphic ones). There are then two non-Abelian ones up to isomorphism, D_4 and the quaternion group Q of Example 3.3.7; you show this in Exercise 7.5 #11 above. So there are 5 total.

For order 15, see Proposition 7.4.6. This is the only case in this exercise that really uses the Sylow theorems. The Sylow theorems generally play a much more crucial role when classifying groups of larger orders.

Remark. We exclude $n = 12$ *because it is a little harder. The only non-Abelian groups of order* 12 *are* D_6, A_4, *and a group* T *with elements* $\{a^i b^j | 0 \leq i \leq 2, 0 \leq j \leq 3\}$ *where* a *is an element of order* 3, b *is an element of order* 4, *and* $ba = a^2 b$. *Maybe we will have time to say something about groups of order* 12 *in class. We also stop at* 15 *in this exercise because the classification of groups of order* 16 *is much more complicated: there are* 14 *different groups of order* 16 *up to isomorphism.*