Math 100a Fall 2009 Homework 8

Due 11/20/09 in class or by 4pm in the HW box on the 6th floor of AP&M

Reading

Reading: 3.7, 3.8 (consider reading these sections again), 7.2, 7.3

Assigned Problems

Write up neat solutions to these problems:

Section 3.8: 8 (here $aHa^{-1} = \{aha^{-1} | h \in H\}$, just as on the midterm), 9, 10, 14, 22 (for this one, I suggest finding an appropriate *homomorphism* from $\mathbb{R}^\times$ to the group of positive reals and then applying the fundamental homomorphism theorem to get the isomorphism you want.)

Section 7.2: 9 (follow the example of Example 7.2.5) , 12(b) (in other words, just write down exactly what the class equation (Theorem 7.2.6) says for the group S_5).

Problems not from the text (also to be handed in):

1. Consider $\phi : \mathbb{Z}_{15}^\times \to \mathbb{Z}_{15}^\times$ defined by $\phi([x]_{15}) = [x^2]_{15}$. Show that ϕ is a homomorphism. Find $K = \ker \phi$ and $\operatorname{Im} \phi$, and write down what the fundamental homomorphism theorem says in this case. Write down explicitly the left cosets of K and draw a picture (as we have been doing in class) of what the function ϕ looks like.

2. Let G be any group (so you must use multiplicative notation for its product.) Let $\mathbb{Z} \times \mathbb{Z}$ be the direct product of two copies of $(\mathbb{Z}, +)$.

(a). Let $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ be a homomorphism. Show that what ϕ does to all elements is completely determined once you know what $\phi(0,1)$ and $\phi(1,0)$ are.

(b). Given $a, b \in G$, find a simple condition on a and b which completes the following theorem statement:

Theorem There exists a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ such that $\phi(0,1) = a$ and $\phi(1,0) = b$ if and only if ...

Then prove the theorem.

3. Let $\mathbb{C}$ be the set of complex numbers. Recall that the polar form of a complex number $re^{\theta i}$, where r, θ are real numbers with $r \geq 0$, is the point in the complex plane at a distance r from the origin and making an angle of θ with the positive real axis (measured counterclockwise). In formulas, $re^{i\theta} = r\cos\theta + (r\sin\theta)i$. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the group of nonzero complex numbers under multiplication. Recall that the norm of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$, and that for complex numbers z_1, z_2, one has $|z_1 z_2| = |z_1||z_2|$. (This is covered in Math 20b. If you are very rusty on complex numbers, for example if the term "complex plane" does not ring a bell, there is a review in Appendix A.5.)

(a). Let U be the *circle group* $U = \{z \in \mathbb{C} | |z| = 1\}$. This is the set of points in the complex plane which lie on the unit circle. Prove that U is a subgroup of $\mathbb{C}^\times$.

(b). Prove that $\phi : (\mathbb{R}, +) \to (\mathbb{C}^\times, \cdot)$ defined by $a \mapsto e^{2\pi a i}$ is a homomorphism of groups. Find the kernel and image of this homomorphism.

(c). Prove that $\mathbb{R}/\mathbb{Z} \cong U$.

Remark: visually, one can interpret the homomorphism in (b) as the real line "wrapping" infinitely many times around the unit circle.

4. Let $D_n = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$ be the dihedral group of order $2n$. Find, with proof, all of the conjugacy classes in D_n. Write down exactly what the class equation (Theorem 7.2.6) says for this particular group. (The answer depends on whether n is even or odd.)