# Math 100a Fall 2009 Homework 7

Due 11/13/09 in class or by 4pm in the HW box on the 6th floor of AP&M

## Reading

Reading: 3.7, 3.8, 7.2.

## Extra Problems

More problems (usually more straightforward ones) for extra practice. Don't hand in.

Section 3.7. 1(a), 3, 5, 6, 11, 12, 13,15, 18.

Section 3.8 1(a), 2, 3, 7, 8

## Assigned Problems

Write up neat solutions to these problems:

Section 3.7: 1(b), 7(b)(d), 9 (note these are onto (surjective) homomorphisms), 14, 16 (personally, I find the book's hint complicated. I prefer to think about why the left and right cosets must be the same.)

Section 3.8: 1(b), 4 (here the notation refers to Example 3.6.3), 11.

Problems not from the text (also to be handed in):

1. Let $(\mathbb{Q}, +)$ be the group of rational numbers $\{a/b \,|a, b \in \mathbb{Z}, b \neq 0\}$ with addition. Clearly $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$, and it is a normal subgroup because $\mathbb{Q}$ is Abelian. So we can consider the factor group $G = \mathbb{Q}/\mathbb{Z}$. This is a group with rather weird properties, as you are about to show.

(a). Given an element in $G$ (a coset $(a/b) + \mathbb{Z}$), find the order of that element. Show then that every element of $G$ has finite order, and that for every positive $n > 0$, $G$ contains an element of order $n$.

(b). Show that $G$ is an infinite group. So $G$ is an infinite group such that all of its elements have finite order.

2. Let $G = S_4$, the symmetric group of permutations of $\{1, 2, 3, 4\}$. We know that $|G| = 4! = 24$.

(a). Prove that $H = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of $G$. (Hint: showing $H$ is normal can be done directly from the definition, but it is quite tedious. It might help to recall Exercise 2.3 #13: it shows the conjugate of a $k$-cycle is also a $k$-cycle. The same idea can be used to show if an element is a product of two disjoint 2-cycles when written in disjoint cycle representation, then any conjugate will also be a product of two disjoint 2-cycles.)

(b). By part (a), the factor group $G/H$ makes sense. Since $|H| = 4$, the group $G/H$ has $24/4 = 6$ elements. Thus by exercise 1 on last week's homework, $G/H$ must be isomorphic to either $\mathbb{Z}_6$ or to $S_3$. Which one is it?

3. Let $G$ be any group (so you must use multiplicative notation for its product.) Let $\mathbb{Z} \times \mathbb{Z}$ be the direct product of two copies of $(\mathbb{Z}, +)$.

(a). Let $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ be a homomorphism. Show that what $\phi$ does to all elements is completely determined once you know what $\phi(0, 1)$ and $\phi(1, 0)$ are.

(b). Given $a, b \in G$, find necessary and sufficient conditions for there to exist a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ such that $\phi(0, 1) = a$ and $\phi(1, 0) = b$.