

Math 100a Fall 2009 Homework 5

Due 10/30/09 in class or by 4pm in the HW box on the 6th floor of AP&M

Reading

All references are to Beachy and Blair, 3rd edition.

Reading: 3.3 (concentrate on the material on direct products of groups), 3.4, 3.5.

Assigned Problems

Write up neat solutions to these problems:

Section 3.3: 5, 8

Problems not from the text (also to be handed in):

1. Let m and n be positive integers. If $\gcd(m, n) = 1$ then we proved in class that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$. Since \mathbb{Z}_{mn} is a cyclic group, this shows that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic in this case.

In this problem I want you to show that if $\gcd(m, n) > 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n$ is *not* cyclic.

2. Let m and n be positive integers with $\gcd(m, n) = 1$. You proved in an earlier exercise (1.4 #29) that the function

$$\begin{aligned} f : \mathbb{Z}_{mn}^\times &\rightarrow \mathbb{Z}_m^\times \times \mathbb{Z}_n^\times \\ [a]_{mn} &\mapsto ([a]_m, [a]_n) \end{aligned}$$

is a well-defined bijection of sets.

We know now that \mathbb{Z}_m^\times is a group under multiplication, and we also know how to interpret $\mathbb{Z}_m^\times \times \mathbb{Z}_n^\times$ as the direct product of two groups. In this problem, I want you to prove that f is an isomorphism of groups, and conclude that $\mathbb{Z}_{mn}^\times \cong \mathbb{Z}_m^\times \times \mathbb{Z}_n^\times$. (You can assume the result of the earlier exercise, that f is a bijection, although if you didn't get that exercise the first time you might want to go over it again.)

Midterm exam

Review session: Your TA will hold a review session on Monday October 26, 5-6:30pm in Center 222.

The midterm exam is next Wednesday in class. It will cover Homeworks 1-4, and sections 1.1-1.4, 2.3, and 3.1-3.2 of the text. (Sections 2.1 and 2.2 are assumed background, so I would not explicitly test you on those sections.)

You should know/understand

1. All homework exercises. If there are any you didn't get, now is a good time to go back and look at them again.
2. All definitions we have covered.
3. The statements of and general idea of the proofs of theorems we have covered. For example, know the outline of the proof of Lagrange's theorem.
4. The list of examples of groups we have covered and their basic properties. This includes: groups of numbers with addition; groups of numbers with multiplication; S_n and A_n ; $(\mathbb{Z}_n, +)$, $(\mathbb{Z}_n^\times, \cdot)$, matrix groups.

One question will be "short answers", where no formal proof is required. The other problems (2-3 of them) will all involve proofs rather than computations.

Sample problems.

1. (short answers)
 - (a). Give an example of an infinite non-Abelian group.
 - (b). What is the order of $[4]_7$ in the group \mathbb{Z}_7^\times ?
 - (c). What is the order of the permutation $\sigma = (12)(245)$ in the group S_5 ?
 - (c). Describe all subgroups of $(\mathbb{Z}_{17}, +)$.
2. State Lagrange's theorem. Then give a brief (5-10 sentence) outline of the proof. For example, if the proof involves some equivalence relation, just say that one proves it is an equivalence relation, don't actually prove it.
3. Let G be a group, and let $a \in G$ be an element of order n for some positive integer n . Prove that for any positive integer d such that $d|n$, there exists an element in G of order d .