

Math 100a Fall 2009 Homework 4

Due 10/23/09 in class or by 4pm in the HW box on the 6th floor of AP&M

Reading

All references are to Beachy and Blair, 3rd edition.

Reading: 2.3, 3.3, 3.4

Warmup problems

These are easier/extra problems which you can look at for extra practice if you want. Do not hand these in.

Section 2.3: 1, 2, 4, 5, 6, 7, 8, 9

Assigned Problems

Write up neat solutions to these problems:

Section 3.2: 24

Section 2.3: 3, 10, 13

Problems not from the text (also to be handed in):

1. Suppose that G is a group of order 91 and $a \in G$. You are given that G is *not* cyclic, and that $a^{52} \neq e$. Find $o(a)$.

2. Let S_n be the symmetric group of permutations of $\{1, 2, 3, \dots, n\}$, for some $n \geq 3$. Recall that the *center* $Z(G)$ of a group G is the set of all elements x of G which commute with every other element; in other words, $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$.

Show that $Z(S_n) = \{e\}$.

3. Let G be an *Abelian* group, and let a and b be elements of G which have finite order. For convenience, define $m = o(a) < \infty$ and $n = o(b) < \infty$, and $k = o(ab)$. The point of this problem is to study how the order of ab is related to the order of a and the order of b .

(a). Prove that k is finite and in fact that $k \mid \text{lcm}[m, n]$. (Hint: first explain why $(ab)^i = a^i b^i$ for all positive integers i .)

(b). Show that if $\text{gcd}(m, n) = 1$, then $k = mn$. (Hint: If $a^k b^k = e$, then $a^k = b^{-k}$. What happens when you raise both sides of this equation to a further power?)

Remark: In a non-Abelian group, exercise 2 fails completely; there is in general no relationship at all between $o(a)$, $o(b)$, and $o(ab)$ in a non-Abelian group. The next exercise shows that we can even have a and b of order 2 but $o(ab)$ of *infinite* order!

4. Let G be the group of all permutations of the infinite set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Remember that G is the set of all bijective functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and that the operation in G is composition of functions, i.e. fg means $f \circ g$. Let $f, g \in G$ be the functions given by the formulas $f(x) = -x$ and $g(x) = 1 - x$ (you can take as given that these really are bijective functions and thus belong to G .) Prove that in the group G , $o(f) = 2$ and $o(g) = 2$, but that $o(fg) = \infty$.