Math 100a Fall 2009 Homework 1

Due 10/02/09 in class

Reading

All references are to Beachy and Blair, 3rd edition.

For review in advance of the placement quiz on Monday 9/28, review the basics of sets, functions, equivalence relations, and proof by induction. In particular, review sections A.1, A.4, 2.1 and 2.2 in the text.

The reading assignment for the week ending 10/02 is Sections 1.1-1.4, Section A.2 and Section 3.1.

Some of the "extra problems" on this week's homework are meant to review the concepts of proof by induction, functions, and equivalence relations. The assigned problems are mostly about the properties of integers covered in Section 1.1-1.2.

Extra Problems

These are problems I like which I decided not to assign. You may find it useful to look over these extra problems to check your understanding of the material. They are not to be handed in.

There is nothing wrong of course with doing even more exercises from the text not listed here!

Section A.4: 2, 12.

Section 1.1: 3(e), 5(e), 7, 10, 12, 14, 17, 19, 20

Section 1.2: 14, 16, 20

Section 2.1: 2, 4, 6, 13-19

Section 2.2: 3, 4, 5, 9.

Assigned Problems

Write up neat solutions to these problems:

Section 1.1: 11 (Do not use prime factorizations for this problem), 17.

Section 1.2: 8, 10, 15 (Do not use prime factorizations for these first three problems), 18, 19, 23.

Problems not from the text (also to be handed in):

1. Recall that the Fibonacci numbers are the sequence of numbers $f_1, f_2, f_3, \ldots$ defined by setting $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$. This sequence begins $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.

One thing that is very unclear from this definition is how one would efficiently calculate f_n for some large n. In fact there is a formula for the nth fibonacci number. Prove that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \geq 1$.

Show also that there is an integer n_0 such that for all $n \geq n_0$, f_n is the nearest integer to the number $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$. This gives an easy way of estimating the nth Fibonacci number using a calculator or computer.

2. Let $a, b, c \in \mathbb{Z}$ be given. Show that the equation $ax + by = c$ has a solution with $x, y \in \mathbb{Z}$ if and only if $\gcd(a, b) \mid c$. Supposing the equation does have a solution and we know one, say $x = x_0, y = y_0$, show that there are infinitely many solutions with $x, y \in \mathbb{Z}$ and find (with proof) formulas that give all such solutions for x and y in terms of the known solution.

(FYI, in practice one uses the Euclidean algorithm to actually find some solution $x = x_0, y = y_0$; your text explains this in Section 1.1.)

3. Recall from class (or from Example 2.2.3 in the text) that the rational numbers $\mathbb{Q}$ can be constructed from the integers $\mathbb{Z}$ in the following way. Let $T = \mathbb{Z} - \{0\}$ be the set of nonzero integers, and let $S = \mathbb{Z} \times T$. Put an equivalence relation on S where $(a, b) \sim (c, d)$ if $ad = bc$. The rational numbers $\mathbb{Q}$ is defined to be the set of equivalence classes $S/\sim$. Intuitively, the equivalence class $[(a, b)]$ corresponds to the more familiar notation a/b.

Using the definition of $\mathbb{Q}$ as $S/\sim$ as above, show that the rules $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$ and $[(a, b)] \cdot [(c, d)] = [(ac, bd)]$ give well-defined binary operations $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$. Then prove that $\mathbb{Q}$ is a group under the binary operation $+$. Prove also that $\mathbb{Q} - \{[(0, 1)]\}$ is a group under the binary operation $\cdot$.

(In practice, of course, one uses the more intuitive notation a/b for rational numbers rather than thinking in terms of equivalence classes of ordered pairs (a, b).)