MATH 100A FALL 2015 SAMPLE MIDTERM

Instructions: Justify all of your answers. You may quote the theorems that we proved in class, or that are proved in the textbook, in your proofs, unless the problem says otherwise or the intent of the problem is to reprove the theorem. Generally, do not quote the result of a homework exercise in your proof—if you need such a result you should go through the proof again.

1. (a). Let G be a set with binary operation $*$. Define what it means for G to be a group with this operation.

(b). Given an element a in a group G, define the *order* of the element a.

2. Let H be a subgroup of a group G. Let $x \in G$ be any element.
Define $xHx^{-1} = \{xhx^{-1} | h \in H\}$. Prove that xHx^{-1} is a subgroup of G.

3. Let G be the group $\mathbb{Z}_5^\times$ of units modulo 5 under multiplication.
(a). Is G a cyclic group?
(b). Is G isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$?

4. Let G be a group. Prove that if $x^2 = e$ for all $x \in G$, then G is Abelian.

5. Let G be a finite group with subgroup H. Recall that there is an equivalence relation $\sim$ on G such that $a \sim b$ means $ab^{-1} \in H$, and that the equivalence class $[a]$ is the same as the right coset $Ha = \{ha | h \in H\}$. Thus since G is a disjoint union of the distinct equivalence classes, G is a union of disjoint cosets $G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_n$ for some elements $a_1, \ldots, a_n \in G$. Assume the facts above without proof.
(a). Show that $|H| = |Ha|$ for any $a \in G$.
(b). Use part (a) to prove Lagrange's theorem.