MATH 100A FALL 2015 MIDTERM 1

Instructions: Justify all of your answers, and show your work. You may use the result of one part of a problem in the proof of a later part, even if you do not complete the proof of the earlier part. You may quote basic theorems proved in the textbook or in class, unless the point of the problem is to reproduce the proof of such a theorem, or the problem tells you not to. Do not quote the results of homework exercises.

1 (5 pts). Define what it means for a set G with a binary operation $*$ to be a group.

2 (10 pts). Let G be an Abelian group. Let $H = \{a \in G | \, o(a)$ is a finite odd integer$\}$. Prove that H is a subgroup of G.

3. Let G be a group and consider the function $\phi : G \to G$ given by the formula $\phi(x) = x^{-1}$.

(a) (5 pts). Prove that ϕ is one-to-one and onto.

(b) (5 pts). Prove that ϕ is an isomorphism if and only if the group G is Abelian.

4 (10 pts). Let $G = \mathbb{Z}$ be the group of integers under addition. Prove directly that every subgroup of G is of the form $m\mathbb{Z} = \{mq | q \in \mathbb{Z}\}$ for some $m \geq 0$. (Do not quote the theorem that subgroups of cyclic groups are cyclic. Prove it directly, as you did when this was a homework exercise.)

5. For each of the following groups, decide if the group is cyclic or not and justify your answer.

(a) (5 pts). $\mathbb{Z}_9^\times$.

(b) (5 pts). $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Date: October 23, 2015.