Math 100a Fall 2015 Homework 7

Due Wednesday 11/18/2015 by 5pm in HW box in basement of AP&M

No homework will be due on 11/11 (Veteran's day), so you get roughly two weeks for this homework.

Reading

All references are to Beachy and Blair, 3rd edition.

Read Sections 3.7-3.8, and begin to read section 7.2. We will not cover most of Section 7.1.

Assigned Problems (write up full solutions and hand in):

Section 3.7 #1(b), 2, 9

Section 3.8 #2, 4, 6, 7, 9, 11, 22

Problems not from the text (also to be handed in)

A. Let $(\mathbb{Q}, +)$ be the group of rational numbers $\{a/b \,|a, b \in \mathbb{Z}, b \neq 0\}$ with addition. Clearly $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$, and it is a normal subgroup because $\mathbb{Q}$ is Abelian. So we can consider the factor group $G = \mathbb{Q}/\mathbb{Z}$.

Given an element in G (a coset $(a/b) + \mathbb{Z}$), find the order of that element. Show then that every element of G has finite order, and that for every positive $n > 0$, G contains an element of order n. Conclude that G is an infinite group such that all of its elements have finite order.

B. Let $G = S_4$, the symmetric group of permutations of $\{1, 2, 3, 4\}$. We know that $|G| = 4! = 24$.

(a). Prove that $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of G. (Hint: one way is to use the idea of Exercise 2.3 #13 to show that if τ is a product of disjoint 2-cycles, then so is $\sigma\tau\sigma^{-1}$ for any σ).

(b). By part (a), the factor group G/H makes sense. Since $|H| = 4$, the group G/H has $24/4 = 6$ elements. It is known that any group of order 6 must be isomorphic to $\mathbb{Z}_6$ or to S_3 (this was an optional problem on an earlier homework). Which one is G/H isomorphic to?

C. Let $\mathbb{C}$ be the set of complex numbers. Recall that the polar form of a complex number $re^{i\theta}$, where r, θ are real numbers with $r \geq 0$, is the point in the complex plane at a distance r from the origin and making an angle of θ with the positive real axis (measured counterclockwise). In formulas, $re^{i\theta} = r\cos\theta + (r\sin\theta)i$. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the group of nonzero complex numbers under multiplication. Recall that the norm of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$, and that for complex numbers z_1, z_2, one has $|z_1 z_2| = |z_1||z_2|$. Let U be the *circle group* $U = \{z \in \mathbb{C} | |z| = 1\}$. This is the set of points in the complex plane which lie on the unit circle, with the operation of multiplication.

(a). Prove that $\phi : (\mathbb{R}, +) \to (\mathbb{C}^\times, \cdot)$ defined by $a \mapsto e^{2\pi ai}$ is a homomorphism of groups.

(b). Prove that $\mathbb{R}/\mathbb{Z} \cong U$. (Hint: identify the kernel and image in (a) and apply the fundamental homomorphism theorem.)

Optional problem (handing in is not required)

Section 3.8 #27.